SIAM J. CONTROL OPTIM.
Vol. 42, No. 5, pp. 1523–1544

THE TOPOLOGICAL ASYMPTOTIC FOR THE HELMHOLTZ EQUATION*

BESSEM SAMET†, SAMUEL AMSTUTZ†, AND MOHAMED MASMOUDI†

Abstract. The aim of the topological sensitivity analysis is to obtain an asymptotic expansion of a functional with respect to the creation of a small hole in the domain. In this paper such an expansion is obtained for the Helmholtz equation with a Dirichlet condition on the boundary of a circular hole. Some applications of this work to waveguide optimization are presented.

Key words. topological optimization, shape optimization, topological gradient, topological asymptotic, Helmholtz equation, waveguides, adjoint equation

AMS subject classifications. 49Q10, 49Q12, 78A25, 78A40, 78A45, 78A50, 35J05

DOI. S0363012902406801

1. Introduction.

Classical shape optimization methods are based on the perturbation of the boundary of the initial shape. The initial and the final shapes have the same topology. The aim of topological optimization is to find an optimal shape without any a priori assumption about the topology of the structure. Many important contributions in this field are concerned with structural mechanics and, in particular, the minimization of the compliance (external work) subject to a volume constraint. In view of the fact that the optimal structure generally has a large number of small holes, most authors [3, 5, 15] have considered composite material optimization. Using the homogenization theory, Allaire and Kohn [3] exhibit a class of laminated materials with an explicit expression for the optimal material at any point of the structure. The range of application of this approach is quite restricted. For this reason, global optimization techniques like genetic algorithms and simulated annealing are used in order to solve more general problems [26]. Unfortunately, these methods are very slow.

The topological gradient has been introduced by Schumacher [27] to minimize a cost function $j(\Omega) = J(\Omega, u_\Omega)$, where u_Ω is the solution to a PDE defined in the domain Ω. The idea is to create a spherical hole $B(x, \varepsilon)$ of radius ε around a point x in Ω. Generally, an asymptotic expansion of the function j can be obtained in the following form:

$$j(\Omega \setminus \overline{B(x,\varepsilon)}) - j(\Omega) = f(\varepsilon)g(x) + o(f(\varepsilon)). \tag{1.1}$$

The function $f(\varepsilon)$ is positive and tends to zero with ε. We call this expansion the topological asymptotic. To minimize the criterion, we have to create holes where g is negative. The optimality condition $g \geq 0$ in Ω is exactly what Buttazzo and Dal Maso [6] have obtained for the Laplace equation, using a relaxed formulation. The topological gradient $g(x)$ has been computed by Schumacher [27] in the case of compliance minimization with Neumann condition on the boundary of the hole. In the same context, Sokolowski [25] gave some mathematical justifications in the

*Received by the editors April 30, 2002; accepted for publication (in revised form) April 2, 2003; published electronically November 6, 2003.
http://www.siam.org/journals/sicon/42-5/40680.html

†UFR MIG, Université Paul Sabatier and CNRS UMR 5640 MIP, 118 route de Narbonne 31062 Toulouse Cedex 4, France (samet@mip.ups-tlse.fr, amstutz@mip.ups-tlse.fr, masmoudi@mip.ups-tlse.fr).

plane stress case and generalized it to various cost functions. A topological sensitivity framework using an adaptation of the adjoint method and a truncation technique has been introduced in [16] in the case of an homogeneous Dirichlet condition imposed on the boundary of a circular hole. The fundamental property of the adjoint technique is to provide the variation of a function with respect to a parameter by using a solution u_Ω and an adjoint state p_Ω which do not depend on the chosen parameter. From the numerical viewpoint, only two systems have to be solved for obtaining $g(x)$ for all $x \in \Omega$. This observation leads to very efficient numerical algorithms. In [10, 11, 12], the topological sensitivity has been obtained in the contexts of linear elasticity, the Poisson equation, and the Stokes problem with general shape functions and arbitrary shaped holes. These publications are concerned with PDE operators whose symbols are homogeneous polynomials.

In this paper, we are interested in the differential operator

$$P = \sum_{i=1}^{2} \frac{\partial^2}{\partial {x_i}^2} + k^2,$$

whose symbol is not homogenous. First, an adaptation of the adjoint method to the topological context is proposed in section 2 for the operator P. Next, a waveguide problem, the truncation method, and the explicit expression of the topological asymptotic are presented in section 3. Finally, an optimization algorithm and some applications of the topological gradient to waveguide optimization are given in section 4. This work was done in collaboration with Alcatel Space Industries.

2. A generalized adjoint method. In this section, the adjoint method is adapted to topological optimization. Let $\mathcal{V}$ be a fixed complex Hilbert space. For $\varepsilon \geq 0$, let $a_\varepsilon(.,.)$ be a sesquilinear and continuous form on $\mathcal{V}$ and l_ε be a semilinear and continuous form on $\mathcal{V}$. We consider the following assumptions.

Hypothesis 1. There exists a sesquilinear and continuous form δ_a, a semilinear and continuous form δ_l, and a real function $f(\varepsilon) > 0$ defined on $\mathbb{R}^*{}_+$ such that

$$\lim_{\varepsilon \to 0} f(\varepsilon) = 0, \tag{2.1}$$

$$\|a_\varepsilon - a_0 - f(\varepsilon)\delta_a\|_{\mathcal{L}_2(\mathcal{V})} = o(f(\varepsilon)), \tag{2.2}$$

$$\|l_\varepsilon - l_0 - f(\varepsilon)\delta_l\|_{\mathcal{L}(\mathcal{V})} = o(f(\varepsilon)), \tag{2.3}$$

where $\mathcal{L}(\mathcal{V})$ (respectively, $\mathcal{L}_2(\mathcal{V})$) denotes the space of continuous and semilinear (respectively, sesquilinear) forms on $\mathcal{V}$.

Hypothesis 2. There exists a constant $\alpha > 0$ such that

$$\inf_{u \neq 0} \sup_{v \neq 0} \frac{|a_0(u,v)|}{\|u\|_\mathcal{V} \|v\|_\mathcal{V}} \geq \alpha.$$

We say that a_0 satisfies the inf-sup condition.

According to (2.2), there exists a constant $\beta > 0$ (independent of ε) such that

$$\inf_{u \neq 0} \sup_{v \neq 0} \frac{|a_\varepsilon(u,v)|}{\|u\|_\mathcal{V} \|v\|_\mathcal{V}} \geq \beta \quad \forall \varepsilon \geq 0.$$

For $\varepsilon \geq 0$, we suppose that the following problem has one solution: find $u_\varepsilon \in \mathcal{V}$ such that

$$a_\varepsilon(u_\varepsilon, v) = l_\varepsilon(v) \qquad \forall v \in \mathcal{V}. \tag{2.4}$$

According to Hypothesis 2, this solution is unique. We have the following lemma.

LEMMA 2.1. *If Hypotheses* 1 *and* 2 *are satisfied, then*

$$\|u_\varepsilon - u_0\|_{\mathcal{V}} = O(f(\varepsilon)).$$

Proof. It follows from Hypothesis 2 that there exists $v_\varepsilon \in \mathcal{V}, v_\varepsilon \neq 0$, such that

$$\beta\|u_\varepsilon - u_0\|_{\mathcal{V}}\|v_\varepsilon\|_{\mathcal{V}} \leq |a_\varepsilon(u_\varepsilon - u_0, v_\varepsilon)|,$$

which implies

$$\begin{aligned}
&\beta\|u_\varepsilon - u_0\|_{\mathcal{V}}\|v_\varepsilon\|_{\mathcal{V}} \\
&\leq |a_\varepsilon(u_0, v_\varepsilon) - l_\varepsilon(v_\varepsilon)| \\
&= |a_\varepsilon(u_0, v_\varepsilon) - (l_\varepsilon - l_0 - f(\varepsilon)\delta_l)(v_\varepsilon) - l_0(v_\varepsilon) - f(\varepsilon)\delta_l(v_\varepsilon)| \\
&= |(a_\varepsilon(u_0, v_\varepsilon) - a_0(u_0, v_\varepsilon)) - (l_\varepsilon - l_0 - f(\varepsilon)\delta_l)(v_\varepsilon) - f(\varepsilon)\delta_l(v_\varepsilon)| \\
&\leq |a_\varepsilon(u_0, v_\varepsilon) - a_0(u_0, v_\varepsilon) - f(\varepsilon)\delta_a(u_0, v_\varepsilon)| + |l_\varepsilon(v_\varepsilon) - l_0(v_\varepsilon) - f(\varepsilon)\delta_l(v_\varepsilon)| \\
&\quad + f(\varepsilon)(|\delta_a(u_0, v_\varepsilon)| + |\delta_l(v_\varepsilon)|).
\end{aligned}$$

Using Hypothesis 1, we obtain

$$\beta\|u_\varepsilon - u_0\|_{\mathcal{V}}\|v_\varepsilon\|_{\mathcal{V}} \leq \big(o(f(\varepsilon)) + f(\varepsilon)(\|\delta_a\|_{\mathcal{L}_2(\mathcal{V})}\|u_0\|_{\mathcal{V}} + \|\delta_l\|_{\mathcal{L}(\mathcal{V})})\big)\,\|v_\varepsilon\|_{\mathcal{V}}. \qquad \square$$

Consider now a cost function $j(\varepsilon) = J(u_\varepsilon)$, where the functional J satisfies

$$J(u + h) = J(u) + \Re(L_u(h)) + o(\|h\|_{\mathcal{V}}) \qquad \forall u, h \in \mathcal{V}. \tag{2.5}$$

Here, L_u is a linear and continuous form on $\mathcal{V}$. We suppose that the following problem has a unique solution p_0, called the adjoint state: find $p_0 \in \mathcal{V}$ such that

$$a_0(v, p_0) = -L_{u_0}(v) \qquad \forall v \in \mathcal{V}. \tag{2.6}$$

For $\varepsilon \geq 0$, we define the Lagrangian operator $\mathcal{L}_\varepsilon$ by

$$\mathcal{L}_\varepsilon(u, v) = J(u) + a_\varepsilon(u, v) - l_\varepsilon(v) \qquad \forall u, v \in \mathcal{V}.$$

The next theorem gives the asymptotic expansion of $j(\varepsilon)$.

THEOREM 2.2. *If Hypotheses* 1 *and* 2 *are satisfied, then*

$$j(\varepsilon) - j(0) = f(\varepsilon)\Re(\delta_{\mathcal{L}}(u_0, p_0)) + o(f(\varepsilon)), \tag{2.7}$$

where u_0 is the solution to (2.4) *with $\varepsilon = 0$, p_0 is the adjoint state solution to problem* (2.6)*, and*

$$\delta_{\mathcal{L}}(u, v) = \delta_a(u, v) - \delta_l(v) \quad \forall u, v \in \mathcal{V}.$$

Proof. We have that

$$j(\varepsilon) = \mathcal{L}_\varepsilon(u_\varepsilon, v) \qquad \forall \varepsilon \geq 0, \quad \forall v \in \mathcal{V}.$$

Next, choosing $v = p_0$, we obtain

$$\begin{aligned}
j(\varepsilon) - j(0) &= \mathcal{L}_\varepsilon(u_\varepsilon, p_0) - \mathcal{L}_0(u_0, p_0) \\
&= J(u_\varepsilon) - J(u_0) + a_\varepsilon(u_\varepsilon, p_0) - a_0(u_0, p_0) + l_0(p_0) - l_\varepsilon(p_0) \\
&= J(u_\varepsilon) - J(u_0) + \Re(a_\varepsilon(u_\varepsilon, p_0) - a_0(u_0, p_0)) - \Re(l_\varepsilon(p_0) - l_0(p_0)) \\
&= J(u_\varepsilon) - J(u_0) + \Re(a_\varepsilon(u_\varepsilon, p_0) - a_0(u_\varepsilon, p_0) + a_0(u_\varepsilon - u_0, p_0)) \\
&\quad - \Re(l_\varepsilon(p_0) - l_0(p_0) - f(\varepsilon)\delta_l(p_0)) - f(\varepsilon)\Re(\delta_l(p_0)).
\end{aligned}$$

Using (2.5), we have that

$$J(u_\varepsilon) - J(u_0) = \Re(L_{u_0}(u_\varepsilon - u_0)) + o(\|u_\varepsilon - u_0\|_{\mathcal{V}}).$$

Hence,

$$\begin{aligned} &j(\varepsilon) - j(0) \\ &= \Re(a_\varepsilon(u_\varepsilon, p_0) - a_0(u_\varepsilon, p_0)) + \Re(a_0(u_\varepsilon - u_0, p_0) + L_{u_0}(u_\varepsilon - u_0)) + o(\|u_\varepsilon - u_0\|_{\mathcal{V}}) \\ &\quad -\Re(l_\varepsilon(p_0) - l_0(p_0) - f(\varepsilon)\delta_l(p_0)) - f(\varepsilon)\Re(\delta_l(p_0)). \end{aligned}$$

Using that p_0 is the adjoint solution, we obtain

$$\begin{aligned} j(\varepsilon) - j(0) &= \Re(a_\varepsilon(u_\varepsilon, p_0) - a_0(u_\varepsilon, p_0)) + o(\|u_\varepsilon - u_0\|_{\mathcal{V}}) \\ &\quad -\Re(l_\varepsilon(p_0) - l_0(p_0) - f(\varepsilon)\delta_l(p_0)) - f(\varepsilon)\Re(\delta_l(p_0)) \\ &= \Re((a_\varepsilon - a_0)(u_0, p_0)) + \Re((a_\varepsilon - a_0)(u_\varepsilon - u_0, p_0)) + o(\|u_\varepsilon - u_0\|_{\mathcal{V}}) \\ &\quad -\Re(l_\varepsilon(p_0) - l_0(p_0) - f(\varepsilon)\delta_l(p_0)) - f(\varepsilon)\Re(\delta_l(p_0)). \end{aligned}$$

It follows from Hypothesis 1 that

$$\begin{aligned} j(\varepsilon) - j(0) = {}& f(\varepsilon)\Re(\delta_a(u_0, p_0)) + o(f(\varepsilon)) + f(\varepsilon)\Re(\delta_a(u_\varepsilon - u_0, p_0)) + o(f(\varepsilon))\|u_\varepsilon - u_0\|_{\mathcal{V}} \\ &+ o(\|u_\varepsilon - u_0\|_{\mathcal{V}}) - f(\varepsilon)\Re(\delta_l(p_0)). \end{aligned}$$

Finally, from Lemma 2.1 and the hypothesis $\lim_{\varepsilon\to 0} f(\varepsilon) = 0$, we have

$$j(\varepsilon) = j(0) + f(\varepsilon)\Re(\delta_a(u_0, p_0) - \delta_l(p_0)) + o(f(\varepsilon)),$$

since δ_a is continuous by assumption. □

3. A waveguide problem.

In this section, we study a problem of a waveguide as a component of a spatial antenna feeding system. Because the waveguide $\mathcal{O}$ has a uniform thickness, $\mathcal{O} = \Omega\times]a, b[$, $\Omega \subset \mathbb{R}^2$, and the electric field has a vertical polarization (normal to Ω), the three-dimensional problem can be reduced to a two-dimensional problem in Ω, called the H-plane model. We assume that Ω is a domain of $\mathbb{R}^2$ with a regular boundary $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \cdots \cup \Gamma_N$, $N \in \mathbb{N}^*$. We denote by u_Ω the normal component to Ω of the electric field. It is a solution to the Helmholtz problem:

$$\left\{ \begin{array}{lll} \Delta u_\Omega + k^2 u_\Omega & = 0 & \text{in } \Omega, \\ u_\Omega & = 0 & \text{on } \Gamma_0, \\ \partial_n u_\Omega - iku_\Omega & = h_j & \text{on } \Gamma_j, j = 1, 2, \ldots, N, \end{array} \right. \tag{3.1}$$

where $\partial_n u_\Omega$ is the normal derivative of u_Ω, $k \in \{k \in \mathbb{C}^*/\Im(k) \geq 0\}$, and $h_j \in H_{00}^{\frac{1}{2}}(\Gamma_j)'$ for all $j \in \{1, 2, \ldots, N\}$. The first boundary condition means that Γ_0 is a perfect metallic surface. When $h_j = 0$, the last equation is an approximate absorbing boundary condition (the normal incident plane waves are completely absorbed). When $h_j \neq 0$, it is a transmission condition. We prove in section 5.1 that problem (3.1) has one and only one solution in the Hilbert space

$$\mathcal{V}_\Omega = \{u \in H^1(\Omega), u = 0 \text{ on } \Gamma_0\}. \tag{3.2}$$

Here and in the following, all the Sobolev spaces involve complex-valued functions.

For a given $x \in \Omega$, let us consider the perforated open set $\Omega_\varepsilon = \Omega\setminus\overline{B(x, \varepsilon)}$, where x is a point of Ω and $B(x, \varepsilon)$ is the ball of center x and of radius ε (see Figure 1). We

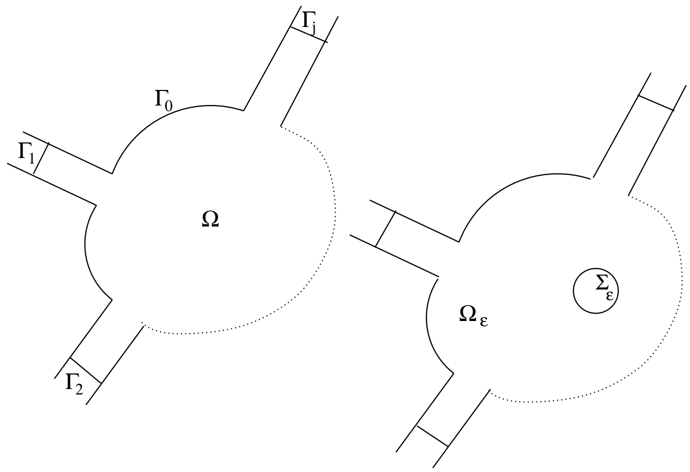


FIG. 1. *The initial domain and the same domain after the perforation.*

assume that $\varepsilon > 0$ is small enough, and we denote $\Sigma_\varepsilon = \partial B(x, \varepsilon)$. Our aim is to get the sensitivity analysis of u_{Ω_ε}, being the unique solution (see section 5.1) to

$$\left\{\begin{array}{lll} \Delta u_{\Omega_\varepsilon} + k^2 u_{\Omega_\varepsilon} & = 0 & \text{in } \Omega_\varepsilon, \\ u_{\Omega_\varepsilon} & = 0 & \text{on } \Gamma_0, \\ u_{\Omega_\varepsilon} & = 0 & \text{on } \Sigma_\varepsilon, \\ \partial_n u_{\Omega_\varepsilon} - iku_{\Omega_\varepsilon} & = h_j & \text{on } \Gamma_j, j = 1, 2, \dots, N, \end{array}\right. \tag{3.3}$$

with respect to ε at $\varepsilon = 0$. The solution of problem (3.3) is defined on the variable open set Ω_ε; thus it belongs to a functional space which depends on ε. Hence, if we want to derive the asymptotic expansion of a function of the form

$$j(\varepsilon) = J(u_{\Omega_\varepsilon}), \tag{3.4}$$

we cannot apply directly the tools of section 2, which require a fixed functional space. In classical shape optimization, this requirement can be satisfied with the help of a domain parameterization technique [13, 20, 17]. This technique involves a fixed domain and a bi-Lipshitz map between this domain and the modified one. In the topology optimization context, such a map does not exist between Ω and Ω_ε. However, a functional space independent of ε can be constructed by using a domain truncation technique.

3.1. The domain truncation. Let $R > \varepsilon$ be such that the ball $B(x, R)$ is included in Ω. The boundary of $B(x, R)$ is denoted by Σ_R. The truncated domain $\Omega \backslash \overline{B(x, R)}$ is denoted by Ω_R, and D_ε denotes the corona $B(x, R) \backslash \overline{B(x, \varepsilon)}$ (see Figure 2).

For a $\Psi \in H^{\frac{1}{2}}(\Sigma_R)$, we consider u^ε_Ψ the solution to the problem

$$\left\{\begin{array}{lll} \Delta u^\varepsilon_\Psi + k^2 u^\varepsilon_\Psi & = 0 & \text{in } D_\varepsilon, \\ u^\varepsilon_\Psi & = \Psi & \text{on} \Sigma_R, \\ u^\varepsilon_\Psi & = 0 & \text{on } \Sigma_\varepsilon \end{array}\right. \tag{3.5}$$

and the *Dirichlet-to-Neumann* operator

$$\begin{array}{llll} T^\varepsilon : & H^{1/2}(\Sigma_R) & \longrightarrow & H^{-1/2}(\Sigma_R), \\ & \Psi & \longmapsto & T^\varepsilon \Psi = \nabla u^\varepsilon_\Psi . n_{|\Sigma_R}, \end{array}$$

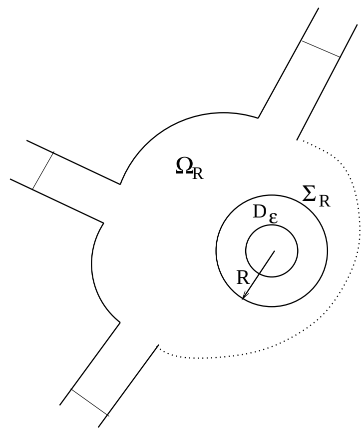


FIG. 2. *The truncated domain.*

where $n_{|\Sigma_R}$ denotes the outward normal to the boundary Σ_R. Using the Poincaré inequality, we obtain that, for $\varepsilon < R < (\sqrt{2}|k|)^{-1}$, problem (3.5) is coercive. Hence it has one and only one solution.

We consider the truncated problem: find u_ε such that

$$\left\{\begin{array}{lll} \Delta u_\varepsilon + k^2 u_\varepsilon & = 0 & \text{in } \Omega_R, \\ u_\varepsilon & = 0 & \text{on } \Gamma_0, \\ \partial_n u_\varepsilon + T^\varepsilon u_\varepsilon & = 0 & \text{on } \Sigma_R, \\ \partial_n u_\varepsilon - iku_\varepsilon & = h_j & \text{on } \Gamma_j, j = 1, 2, \ldots, N. \end{array}\right. \tag{3.6}$$

The variational formulation associated to problem (3.6) is the following: find $u_\varepsilon \in \mathcal{V}_R$ such that

$$a_\varepsilon(u_\varepsilon, v) = l(v) \qquad \forall v \in \mathcal{V}_R, \tag{3.7}$$

where the functional space $\mathcal{V}_R$, the sesquilinear form a_ε, and the semilinear form l are defined by

$$\mathcal{V}_R = \{u \in H^1(\Omega_R), u = 0 \text{ on } \Gamma_0\}, \tag{3.8}$$

$$a_\varepsilon(u, v) = \int_{\Omega_R} \nabla u.\overline{\nabla v}\, dx - k^2 \int_{\Omega_R} u\overline{v}\, dx + \int_{\Sigma_R} (T^\varepsilon u)\overline{v}\, d\gamma(x) \tag{3.9}$$

$$-ik \sum_{j=1}^{N} \int_{\Gamma_j} u\overline{v}\, d\gamma(x), \tag{3.10}$$

$$l(v) = \sum_{j=1}^{N} \int_{\Gamma_j} h_j \overline{v}\, d\gamma(x). \tag{3.11}$$

Here, $\nabla u.\overline{\nabla v} = \sum_{i=1}^{2} \frac{\partial u}{\partial x_i} \frac{\partial \overline{v}}{\partial x_i}$ and $d\gamma(x)$ is the Lebesgue measure on the boundary. The following result is standard in PDE theory.

PROPOSITION 3.1. *Problem* (3.6) *has one and only one solution in* $\mathcal{V}_R$ *which is the restriction to* Ω_R *of the solution to* (3.3).

Proof. Existence: Applying the definition of T^ε, we prove that the restriction to Ω_R of the solution to (3.3) is a solution to (3.6).

Uniqueness: Any solution u to problem (3.6) can be extended in Ω_ε to the solution to problem (3.3): we use the solution u^ε_Ψ to (3.5) with $\Psi = u_{|\Sigma_R}$. $\square$

We have now at our disposal the fixed Hilbert space $\mathcal{V}_R$ required by section 2. We assume that the function J is defined in a neighbor part of Γ. Then we have

$$j(\varepsilon) = J(u_{\Omega_\varepsilon}) = J(u_\varepsilon) \qquad \forall \varepsilon \geq 0. \tag{3.12}$$

3.2. Variation of the sesquilinear form. The variation of the sesquilinear form $a_\varepsilon - a_0$ reads

$$a_\varepsilon(u, v) - a_0(u, v) = \int_{\Sigma_R} \big((T^\varepsilon - T^0)u\big)\, \overline{v}\, d\gamma(x). \tag{3.13}$$

Hence, the problem reduces to the computation of $(T^\varepsilon - T^0)\Psi$ for $\Psi = u_{|\Sigma_R}$.
We have the following proposition.

Proposition 3.2. *The solution u^ε_Ψ to problem* (3.5) *and the operator T^ε are given by the explicit expressions:*

$$u^\varepsilon_\psi(r, \theta) = \sum_{n\in\mathbb{Z}} \frac{J_n(kr)Y_n(k\varepsilon) - J_n(k\varepsilon)Y_n(kr)}{J_n(kR)Y_n(k\varepsilon) - Y_n(kR)J_n(k\varepsilon)} \psi_n e^{in\theta}$$

and

$$T^\varepsilon \psi = k \sum_{n\in\mathbb{Z}} \frac{J_n'(kR)Y_n(k\varepsilon) - J_n(k\varepsilon)Y_n'(kR)}{J_n(kR)Y_n(k\varepsilon) - Y_n(kR)J_n(k\varepsilon)} \psi_n e^{in\theta}, \tag{3.14}$$

where (r, θ) are the polar coordinates in $\mathbb{R}^2$, (Ψ_n) are the Fourier coefficients of Ψ, and (J_n) and (Y_n) are, respectively, the Bessel functions of the first and the second kind.

Proof. We have in polar coordinates

$$u^\varepsilon_\psi(r, \theta) = \sum_{n\in\mathbb{Z}} c_n(r) e^{in\theta},$$

where $c_n(r)$ satisfies the differential equation:

$$\frac{d^2 c_n}{dr^2} + \frac{1}{r}\frac{dc_n}{dr} + \left(k^2 - \frac{n^2}{r^2}\right) c_n(r) = 0 \qquad \forall n \in \mathbb{Z},$$

and thus c_n is a linear combination of J_n and Y_n Bessel functions:

$$c_n(r) = a_n J_n(kr) + b_n Y_n(kr) \quad \forall n \in \mathbb{Z}.$$

Using the boundary conditions, we obtain

$$a_n = \frac{Y_n(k\varepsilon)}{J_n(kR)Y_n(k\varepsilon) - Y_n(kR)J_n(k\varepsilon)} \psi_n, \quad b_n = \frac{-J_n(k\varepsilon)}{J_n(kR)Y_n(k\varepsilon) - Y_n(kR)J_n(k\varepsilon)} \psi_n. \qquad \square$$

In particular, for $\varepsilon = 0$ we have the following proposition.

Proposition 3.3. *The solution u^0_Ψ and the operator T^0 are given by the explicit expressions*

$$u^0_\psi(r, \theta) = \sum_{n\in\mathbb{Z}} \frac{J_n(kr)}{J_n(kR)} \psi_n e^{in\theta}$$

and

$$T^0\psi = k\sum_{n\in\mathbb{Z}} \frac{J_n'(kR)}{J_n(kR)}\psi_n e^{in\theta}, \tag{3.15}$$

where u_ψ^0 *is the solution to* (3.5) *for* $\varepsilon = 0$.

For $\Psi \in H^s(\Sigma_R)$, let

$$\|\psi\|^2_{s,\Sigma_R} = \sum_{n\in\mathbb{Z}} |\psi_n|^2(1+|n|)^{2s} \tag{3.16}$$

be the norm of Ψ in this space. The so defined norm is equivalent to the usual norm of $H^s(\Sigma_R)$. We introduce the operator:

$$\begin{array}{rccc} \delta_T: & H^{1/2}(\Sigma_R) & \longrightarrow & H^{-1/2}(\Sigma_R), \\ & \Psi & \longmapsto & \delta_T\Psi = \frac{1}{RJ_0^2(kR)}\Psi_0. \end{array}$$

We have the following lemma.

Lemma 3.4. *We have that*

$$\left\| T^\varepsilon - T^0 - \frac{-1}{\log(\varepsilon)}\delta_T \right\|_{\mathcal{L}(H^{1/2}(\Sigma_R);H^{-1/2}(\Sigma_R))} = o\left(\frac{-1}{\log(\varepsilon)}\right).$$

Proof. Let $\Psi \in H^{\frac{1}{2}}(\Sigma_R)$. Using the series (3.14) and (3.15), we obtain

$$\begin{aligned}(T^\varepsilon - T^0)\psi &= k\sum_{n\in\mathbb{Z}} \frac{J_n'(kR)Y_n(k\varepsilon) - J_n(k\varepsilon)Y_n'(kR)}{J_n(kR)Y_n(k\varepsilon) - Y_n(kR)J_n(k\varepsilon)}\psi_n e^{in\theta} - k\sum_{n\in\mathbb{Z}}\frac{J_n'(kR)}{J_n(kR)}\psi_n e^{in\theta} \\ &= k\sum_{n\in\mathbb{Z}^*} \frac{J_n'(kR)Y_n(k\varepsilon) - J_n(k\varepsilon)Y_n'(kR)}{J_n(kR)Y_n(k\varepsilon) - Y_n(kR)J_n(k\varepsilon)}\psi_n e^{in\theta} - k\sum_{n\in\mathbb{Z}^*}\frac{J_n'(kR)}{J_n(kR)}\psi_n e^{in\theta} \\ &\quad -k\frac{Y_0'(kR)J_0(kR) - Y_0(kR)J_0'(kR)}{J_0^2(kR)}\frac{J_0(k\varepsilon)J_0(kR)}{J_0(kR)Y_0(k\varepsilon) - Y_0(kR)J_0(k\varepsilon)}\psi_0.\end{aligned}$$

We have that [1]

$$\begin{aligned}\frac{Y_0'(kR)J_0(kR) - Y_0(kR)J_0'(kR)}{J_0^2(kR)} &= \frac{W\{J_0(kR), Y_0(kR)\}}{J_0^2(kR)} \\ &= \frac{2}{\pi kR}\frac{1}{J_0^2(kR)},\end{aligned}$$

where W is the Wronskian. Then

$$\begin{aligned}(T^\varepsilon - T^0)\psi &= k\sum_{n\in\mathbb{Z}^*} \frac{J_n(k\varepsilon)Y_n(kR)}{Y_n(k\varepsilon)J_n(kR) - Y_n(kR)J_n(k\varepsilon)}\left(\frac{J_n'(kR)}{J_n(kR)} - \frac{Y_n'(kR)}{Y_n(kR)}\right)\psi_n e^{in\theta} \\ &\quad - \frac{2}{\pi}\frac{J_0(k\varepsilon)J_0(kR)}{J_0(kR)Y_0(k\varepsilon) - Y_0(kR)J_0(k\varepsilon)}\frac{1}{RJ_0^2(kR)}\psi_0.\end{aligned} \tag{3.17}$$

We have the following formula [1]:

$$Y_0(k\varepsilon) = \frac{2}{\pi}\left(\log\left(\frac{k\varepsilon}{2}\right) + \gamma\right)J_0(k\varepsilon) + \varepsilon\alpha(\varepsilon), \tag{3.18}$$

where γ denotes Euler's constant and $\alpha(\varepsilon) \to 0$ when $\varepsilon \to 0$. We insert (3.18) into (3.17):

$$(T^\varepsilon - T^0)\psi = \varepsilon R_\varepsilon \Psi + \frac{-1}{\log(\varepsilon)}\left(1 + \frac{M}{\log(\varepsilon)} + \varepsilon\theta(\varepsilon)\right)^{-1} \delta_T \Psi,$$

where M is a constant independent of ε, $\theta(\varepsilon) \to 0$ when $\varepsilon \to 0$ and

$$R_\varepsilon \psi = \sum_{n\in\mathbb{Z}^*} \frac{k}{\varepsilon} \frac{J_n(k\varepsilon)Y_n(kR)}{Y_n(k\varepsilon)J_n(kR) - Y_n(kR)J_n(k\varepsilon)} \left(\frac{J_n'(kR)}{J_n(kR)} - \frac{Y_n'(kR)}{Y_n(kR)}\right)\psi_n e^{in\theta}.$$

Then

$$\left(T^\varepsilon - T^0 - \frac{-1}{\log(\varepsilon)}\delta_T\right)\psi = \varepsilon R_\varepsilon \psi + O(1)\left(\frac{-1}{\log(\varepsilon)}\right)^2 \frac{1}{RJ_0^2(kR)}\psi_0.$$

Using (3.16), we have

$$\| R_\varepsilon \psi \|^2_{-\frac{1}{2};\Sigma_R} = \sum_{n\in\mathbb{Z}^*} \frac{|k|^2}{\varepsilon^2} \left|\frac{J_n(k\varepsilon)Y_n(kR)}{Y_n(k\varepsilon)J_n(kR) - Y_n(kR)J_n(k\varepsilon)}\right|^2$$
$$\cdot \left|\frac{J_n'(kR)}{J_n(kR)(1+|n|)} - \frac{Y_n'(kR)}{Y_n(kR)(1+|n|)}\right|^2 (1+|n|)|\psi_n|^2.$$

Let us prove that there exists a constant $c > 0$ (independent of Ψ and ε) such that for all $0 < \varepsilon < \varepsilon_0 < R$,

$$\|R_\varepsilon \psi \|_{-\frac{1}{2};\Sigma_R} \leq c \| \psi\|_{\frac{1}{2};\Sigma_R}.$$

We have [1]

$$\frac{1}{1+|n|}\frac{J_n'(kR)}{J_n(kR)} = -\frac{1}{1+|n|}\frac{J_{n+1}(kR)}{J_n(kR)} + \frac{n}{1+|n|}\frac{1}{kR}$$

and for $n \to \infty$

$$J_n(z) \sim (2\pi n)^{-\frac{1}{2}} \left(\frac{ez}{2n}\right)^n.$$

Then

$$\lim_{n\to\infty} \frac{1}{1+|n|}\frac{J_{n+1}(kR)}{J_n(kR)} = 0$$

and

$$\left|\frac{1}{1+|n|}\frac{J_n'(kR)}{J_n(kR)}\right| \leq c \quad \forall n \in \mathbb{Z}^*.$$

Here and in what follows, c is a positive constant independent of the data (e.g., of ε and n). Similarly, we have

$$\left|\frac{1}{1+|n|}\frac{Y_n'(kR)}{Y_n(kR)}\right| \leq c \quad \forall n \in \mathbb{Z}^*.$$

Hence,

$$\left|\frac{J_n'(kR)}{J_n(kR)(1+|n|)} - \frac{Y_n'(kR)}{Y_n(kR)(1+|n|)}\right| \leq c \quad \forall n \in \mathbb{Z}^*.$$

We denote

$$f_n(\varepsilon) = \frac{1}{\varepsilon}\left|\frac{J_n(k\varepsilon)Y_n(kR)}{Y_n(k\varepsilon)J_n(kR) - Y_n(kR)J_n(k\varepsilon)}\right|.$$

We have also

$$f_n(\varepsilon) = \left|\frac{\varepsilon J_n(kR)Y_n(k\varepsilon)}{J_n(k\varepsilon)Y_n(kR)} - \varepsilon\right|^{-1}.$$

We show in section 5.3 that there exist n_0 and ε_0 such that

$$\left|\varepsilon\frac{J_n(kR)}{J_n(k\varepsilon)}\right| \geq c\left(\frac{R}{\varepsilon}\right)^{n-1} \qquad \forall n \geq n_0, \quad \forall \varepsilon < \varepsilon_0 \tag{3.19}$$

and

$$\left|\frac{Y_n(k\varepsilon)}{Y_n(kR)}\right| \geq c\left(\frac{R}{\varepsilon}\right)^{n} \qquad \forall n \geq n_0, \quad \forall \varepsilon < \varepsilon_0. \tag{3.20}$$

Using (3.19) and (3.20), we obtain

$$\left|\frac{\varepsilon Y_n(k\varepsilon)J_n(kR)}{J_n(k\varepsilon)Y_n(kR)}\right| \geq c \qquad \forall n \geq n_0, \quad \forall \varepsilon < \varepsilon_0$$

and

$$f_n(\varepsilon) \leq c \qquad \forall n \geq n_0, \quad \forall \varepsilon < \varepsilon_0.$$

For $p \in \{1, 2, \ldots, n_0 - 1\}$, we have $f_p(\varepsilon) \to 0$ when $\varepsilon \to 0$. Then

$$f_n(\varepsilon) \leq c \quad \forall n \in \mathbb{Z}^*, \quad \forall \varepsilon < \varepsilon_0.$$

Hence

$$\|R_\varepsilon \psi\|_{-\frac{1}{2},\Sigma_R} \leq c \parallel \psi \parallel_{\frac{1}{2};\Sigma_R} \quad \forall \psi \in H^{\frac{1}{2}}(\Sigma_R).$$

This completes the proof. ∎

From this lemma we obtain the following proposition.

PROPOSITION 3.5. *Let δ_a be the sesquilinear and continuous form defined on $\mathcal{V}_R$ by*

$$\delta_a(u, v) = \frac{u^{mean}}{J_0(kR)}\frac{\overline{v^{mean}}}{J_0(kR)},$$

where u^{mean} and v^{mean} denote, respectively, the mean values of u and v on Σ_R. We have

$$\left|a_\varepsilon(u, p) - a_0(u, p) - \frac{-2\pi}{\log(\varepsilon)}\delta_a(u, p)\right| = o\left(\frac{-1}{\log(\varepsilon)}\right)\|u\|_{\mathcal{V}_R}\|p\|_{\mathcal{V}_R} \qquad \forall u, p \in \mathcal{V}_R.$$

3.3. The asymptotic expansion. We prove in section 5.2 that the sesquilinear form a_0 satisfies Hypothesis 2 (inf-sup condition).
The adjoint problem is the following: find $p_\Omega \in \mathcal{V}_\Omega$ such that

$$\int_\Omega (\nabla v.\overline{\nabla p_\Omega} - k^2 v\overline{p_\Omega})\, dx - ik\sum_{j=1}^N \int_{\Gamma_j} v\overline{p_\Omega}\, d\gamma(x) = -L_{u_\Omega}(v) \quad \forall v \in \mathcal{V}_\Omega. \tag{3.21}$$

This problem has one and only one solution (see section 5.1). If $L_{u_\Omega} \in H_{00}^{\frac{1}{2}}(\Gamma_m)'$, $m \in \{1, 2, \dots, N\}$, the strong formulation of problem (3.21) is

$$\begin{cases} \Delta p_\Omega + \overline{k}^2 p_\Omega & = 0 & \text{in } \Omega, \\ p_\Omega & = 0 & \text{on } \Gamma_0, \\ \partial_n p_\Omega + i\overline{k} p_\Omega & = -L_{u_\Omega} & \text{on } \Gamma_m, \\ \partial_n p_\Omega + i\overline{k} p_\Omega & = 0 & \text{on } \Gamma_j, j \in \{1, 2, \dots, N\}\backslash\{m\}. \end{cases} \tag{3.22}$$

Hence, all the assumptions of section 2 are satisfied and we can apply the adjoint method. Then we have the following theorem.

THEOREM 3.6. *The function j has the following asymptotic expansion:*

$$j(\varepsilon) - j(0) = \frac{-2\pi}{\log(\varepsilon)} \Re(u_\Omega(x)\overline{p_\Omega}(x)) + o\left(\frac{-1}{\log(\varepsilon)}\right).$$

Proof. Using Theorem 2.2, we obtain

$$j(\varepsilon) - j(0) = \frac{-2\pi}{\log(\varepsilon)} \Re(\delta_a(u_0, p_0)) + o\left(\frac{-1}{\log(\varepsilon)}\right),$$

where u_0 is the solution to (3.7) for $\varepsilon = 0$ and p_0 is the solution to the adjoint problem

$$a_0(v, p_0) = -L_{u_0}(v) \qquad \forall v \in \mathcal{V}_R. \tag{3.23}$$

As observed in Proposition 3.1, u_0 is the restriction to Ω_R of u_Ω. Let us prove that the same property holds for p_0 and p_Ω. For $v \in \mathcal{V}_\Omega$, we denote by p_R and v_R the restriction of p_Ω and v to Ω_R. On the one hand, we have

(3.24)

$$\begin{aligned} &\int_\Omega (\nabla v.\overline{\nabla p_\Omega} - k^2 v\overline{p_\Omega})\, dx - ik\sum_{j=1}^N \int_{\Gamma_j} v\overline{p_\Omega}\, d\gamma(x) \\ &= \int_{\Omega_R} (\nabla v_R.\overline{\nabla p_R} - k^2 v_R\overline{p_R})\, dx - ik\sum_{j=1}^N \int_{\Gamma_j} v_R\overline{p_R}\, d\gamma(x) + \int_{D_0} (\nabla v.\overline{\nabla p_\Omega} - k^2 v\overline{p_\Omega})\, dx \\ &= \int_{\Omega_R} (\nabla v_R.\overline{\nabla p_R} - k^2 v_R\overline{p_R})\, dx - ik\sum_{j=1}^N \int_{\Gamma_j} v_R\overline{p_R}\, d\gamma(x) + \int_{\Sigma_R} (T^0 v_R)\overline{p_R}\, d\gamma(x) \\ &= a_0(v_R, p_R). \end{aligned}$$

On the other hand, due to the fact that J is defined in a neighbor part of Γ, we have that $J(u) = J(u_R)$ for all $u \in \mathcal{V}_\Omega$. Hence

$$L_{u_\Omega}(v) = L_{u_0}(v_R). \tag{3.25}$$

Then, gathering (3.24), (3.21), and (3.25), we obtain

$$a_0(v_R, p_R) = -L_{u_0}(v_R) \qquad \forall v_R \in \mathcal{V}_R,$$

which proves that p_R is the solution to (3.23). Then p_0 is the restriction to Ω_R of p_Ω. It remains to prove that $\delta_a(u_{\Omega|\Omega_R}, p_{\Omega|\Omega_R}) = u_\Omega(x).p_\Omega(x)$. Using that u_Ω is the solution to the Helmholtz equation in the ball $B(x, R)$, we obtain

$$u_\Omega(x) = \frac{u_{\Omega|\Sigma_R}^{mean}}{J_0(kR)}.$$

Similarly, we have

$$\overline{p}_\Omega(x) = \frac{\overline{p_{\Omega|\Sigma_R}^{mean}}}{J_0(kR)}.$$

Hence

$$\begin{aligned}\delta_a(u_0, p_0) &= \delta_a(u_{\Omega|\Omega_R}, p_{\Omega|\Omega_R}) \\ &= u_\Omega(x)\overline{p_\Omega(x)}.\end{aligned}$$

This completes the proof. □

Then the topological gradient is

$$g = \Re(u_\Omega \overline{p_\Omega}).$$

4. Numerical results.

4.1. T-shaped waveguide. We use the topological gradient to design an H-plane T-shaped waveguide. The geometric constraints are shown in Figure 3(a). The input Γ_1 is excited by the TE10 mode (see the second boundary condition of (4.1)): the excitation is given by

$$u_e(y) = \cos\left(\frac{\pi y}{d}\right) \qquad \forall y \in \Gamma_1.$$

We follow the two ideas [22]:

- the initial guess is the free space;
- instead of minimizing the reflected energy, we maximize the transmitted energy on Γ_2 and Γ_3.

At the beginning, only the input and output channels have metallic boundaries. In order to use the finite element method, the design domain is delimited by a fictitious boundary Γ_4 on which an absorbing condition is imposed (see Figure 3(b)). The problem is modelized as follows:

$$(4.1) \qquad \begin{cases} \Delta u + k^2 u &= 0 & \text{in } \Omega, \\ u &= 0 & \text{on } \Gamma_0, \\ \partial_n u - ik'u &= \partial_n u_e - ik'u_e & \text{on } \Gamma_1, \\ \partial_n u - ik'u &= 0 & \text{on } \Gamma_2, \Gamma_3, \\ \partial_n u - iku &= 0 & \text{on } \Gamma_4, \end{cases}$$

where $k^2 = k'^2 + \frac{\pi^2}{d^2}$, d being the length of Γ_1. The perfect conduction on the metallic boundary leads to the first boundary condition $u = 0$ on Γ_0. The third boundary condition prevents reflections on Γ_2, Γ_3. The last equation is an approximate absorbing boundary condition. Here and in the following, we take $k = 10$.

The cost function to maximize is

$$J(u) = |S_{12}(u)|^2 + |S_{13}(u)|^2,$$

where $S_{1j}(u)$ is given by

$$S_{1j}(u) = \int_{\Gamma_j} u_{|\Gamma_j} \cos\left(\frac{\pi x}{d}\right)\, dx, \quad j \in \{2, 3\}.$$

The adjoint state is the solution to

$$\text{(4.2)} \qquad \begin{cases} \Delta \bar{p} + k^2 \bar{p} & = 0 & \text{in } \Omega, \\ \bar{p} & = 0 & \text{on } \Gamma_0, \\ \partial_n \bar{p} - ik'\bar{p} & = 0 & \text{on } \Gamma_1, \\ \partial_n \bar{p} - ik'\bar{p} & = -2\overline{S_{12}(u)} \cos\left(\frac{\pi x}{d}\right) & \text{on } \Gamma_2, \\ \partial_n \bar{p} - ik'\bar{p} & = -2\overline{S_{13}(u)} \cos\left(\frac{\pi x}{d}\right) & \text{on } \Gamma_3, \\ \partial_n \bar{p} - ik\bar{p} & = 0 & \text{on } \Gamma_4. \end{cases}$$

Then the topological gradient is $g = \Re(u\overline{p})$ (see Figure 4(b)). We are interested in the relative loss of energy

$$P(u) = \frac{E_e - (E_2 + E_3)(u)}{E_e},$$

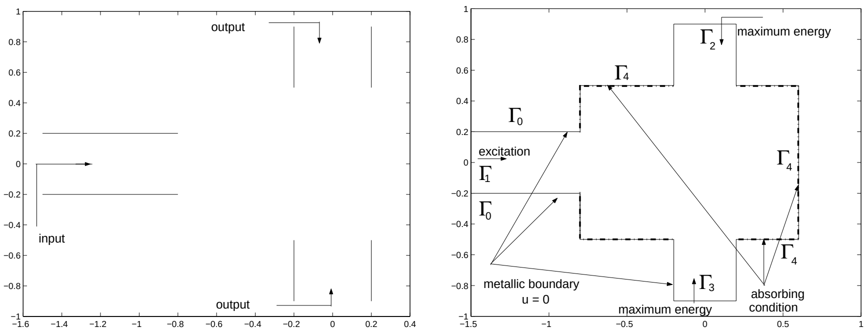
where E_e is the entering energy and $E_j(u)$ is the outgoing energy through Γ_j, $j \in \{2, 3\}$.

We present here the topological optimization procedure. The underlying idea is the following: in the ℓth step of the process, if $\overline{x}$ is such that the topological gradient is higher than a certain value t_ℓ, we insert at this point a Dirichlet node (metal). The constant t_ℓ is chosen by the user, which allows him to take into account other constraints, for example the feasibility. The process is stopped when the topological gradient is everywhere negative in the design domain or when the shape suits the designer. The algorithm is as follows.

- Initialization: choose the initial domain Ω_0, and set $\ell = 0$. The domain Ω_0 is meshed and it is identified with the set of the nodes: $\Omega_0 = \{x_k, k \in \{1, 2, \ldots, n\}\}$. The grid is fixed during the process.
- Repeat:
  1. compute u_ℓ, p_ℓ the direct and adjoint solutions in the domain Ω_ℓ,
  2. compute the topological gradient $g_\ell = \Re(u_\ell \overline{p_\ell})$,
  3. set $\Omega_{\ell+1} = \Omega_\ell \backslash \{x_k, g_\ell(x_k) \geq t_{\ell+1}\}$,
  4. $\ell \leftarrow \ell + 1$.

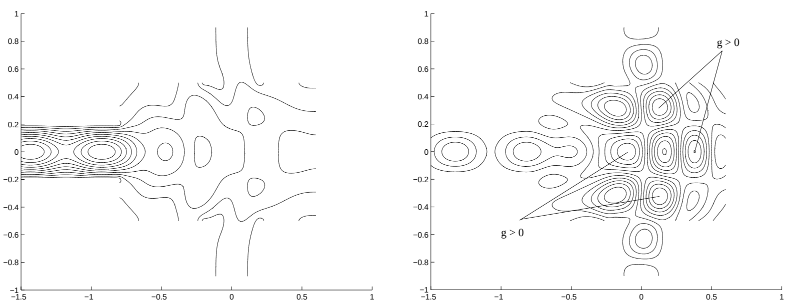
Figure 4 shows the isovalues of $|u|$ and the topological gradient for the initial geometry. In this case, 94.4% of the energy is lost. After two iterations, the loss is reduced to 2.02% (see Figure 5) and the topological gradient is everywhere negative. The last step consists of smoothing the boundary of the domain by inserting some metal where $|u|$ is close to zero. The loss of energy of this waveguide is equal to 1.5% (see Figure 6). The convergence history is given by Figure 7.

4.2. L-shaped waveguide. Here, we use the topological gradient like a decision help system to build a junction between two rectangular waveguides. The initial



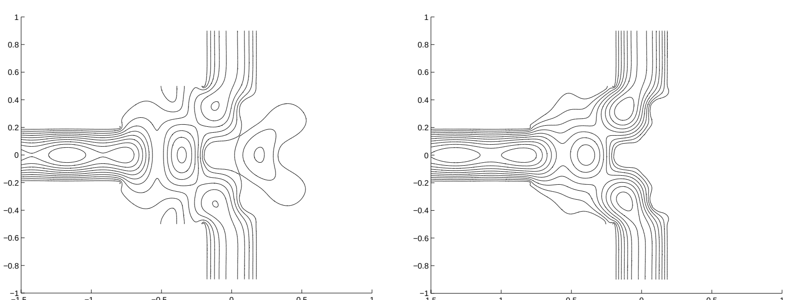
(a) (b)

FIG. 3. *The initial geometry* (a) *and the design domain* (b).



(a) (b)

FIG. 4. *Modulus of the electric field* (a) *and topological gradient* (b).



0

(a) (b)

FIG. 5. *Modulus of the electric fields obtained after a first iteration* (a) *and after two iterations* (b).

geometry and the design domain are given by Figure 8. The cost function to maximize is

$$J(u) = |S_{12}(u)|^2.$$

Figure 9(a) shows the isovalues of $|u|$ for the initial geometry. In this case, 95.43% of the energy is lost. We observe that the topological gradient is high on a quarter

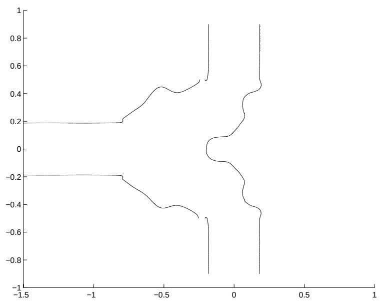


Fig. 6. *Final geometry.*

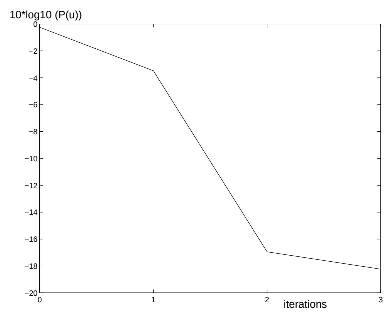
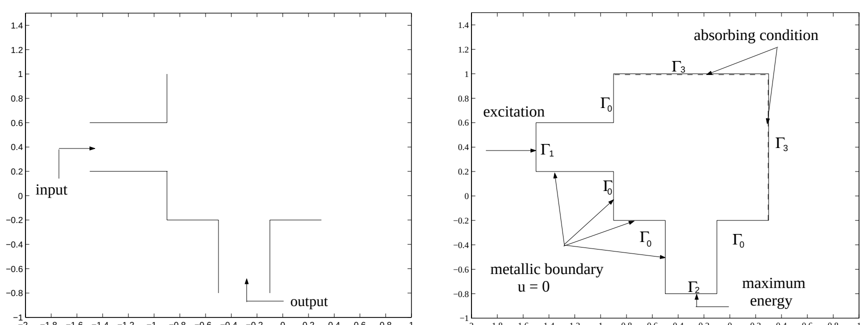


Fig. 7. *Convergence history.*



(a) (b)

Fig. 8. *The initial geometry* (a) *and the design domain* (b).

of circle where we decide to put metal (see Figure 9(b)). The loss of energy of the obtained waveguide is now equal to 0.34% (see Figure 10).

4.3. U-shaped waveguide. Here, the initial guess is a metallic cavity. The geometry of the waveguide is shown in Figure 11. The cost function to maximize is

$$J(u) = |S_{12}(u)|^2.$$

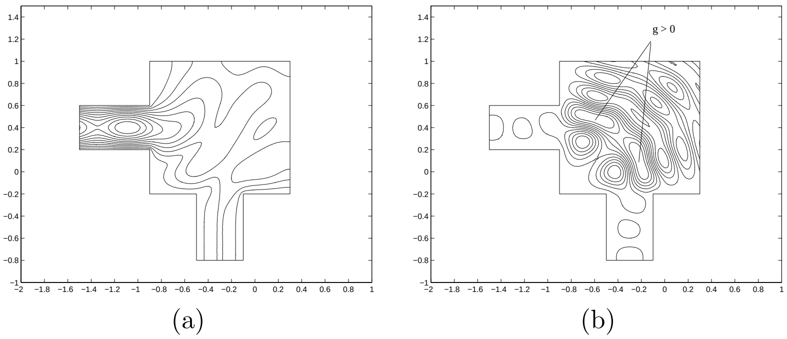


FIG. 9. *Modulus of the electric field* (a) *and topological gradient* (b).

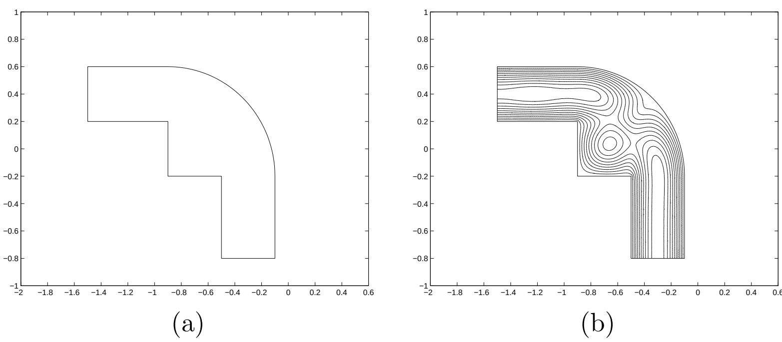


FIG. 10. *Final geometry* (a) *and modulus of the electric field* (b).

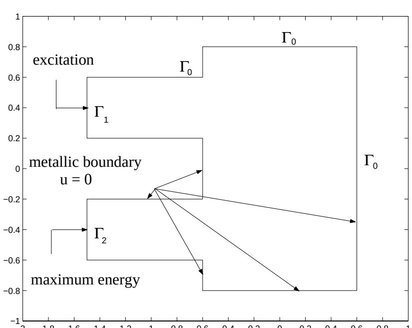


FIG. 11. *Geometry of the initial guide.*

Figure 12(a) shows the isovalues of $|u|$ for the initial geometry. In this case, 88.45% of the energy is reflected. There are three local maximas of the topological gradient (see Figure 12(b)). At each local maxima, we introduce a pointwise Dirichlet condition (a metallic plot). The new energy distribution is shown in Figure 13(a). The loss of energy is now equal to 39.19%. A new analysis is performed: after the introduction of another metallic plot, we obtain the design of Figure 13(b). The objective is fulfilled; the loss of energy is equal to 0.7%. For feasibility reasons, we decide not to insert additional plots.

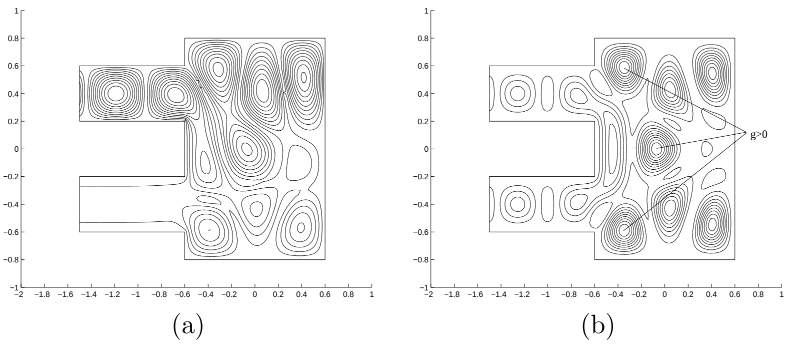


FIG. 12. *Modulus of the electric field* (a) *and topological gradient* (b).

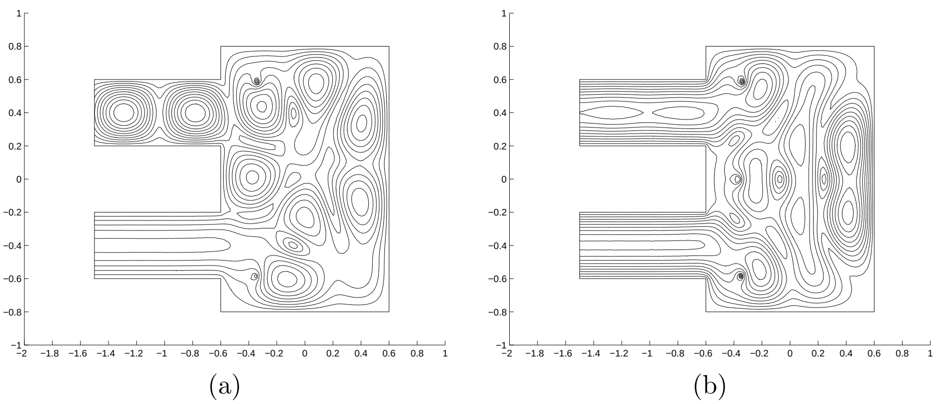


FIG. 13. *Modulus of the electric fields obtained after a first iteration* (a) *and after two iterations* (b).

5. Appendix.

5.1. Existence and uniqueness of the solution. Here we establish the existence and uniqueness of the solution to problem (3.1). Replacing Ω with Ω_ε, the argumentation would be the same for problem (3.3). Without any loss of generality, we suppose here that $N = 1$. The variational form of problem (3.1) is the following: find $u \in \mathcal{V}_\Omega$ satisfying

$$a(u, v) = l(v) \quad \forall v \in \mathcal{V}_\Omega, \tag{5.1}$$

where the functional space $\mathcal{V}_\Omega$, the sesquilinear form a, and the semilinear form l are defined by

$$\begin{aligned} \mathcal{V}_\Omega &= \{v \in H^1(\Omega), v = 0 \text{ on } \Gamma_0\}, \\ a(u, v) &= \int_\Omega (\nabla u.\overline{\nabla v} - k^2 u\overline{v})\, dx - ik \int_{\Gamma_1} u\overline{v}\, d\gamma(x), \\ l(v) &= \int_{\Gamma_1} g\overline{v}\, d\gamma(x). \end{aligned}$$

We split a in the following form:

$$a(u, v) = b(u, v) + c(u, v), \tag{5.2}$$

where b and c are defined by

$$b(u,v) = \int_\Omega (\nabla u.\overline{\nabla v} + u\overline{v})\,dx, \tag{5.3}$$

$$c(u,v) = -(1+k^2)\int_\Omega u\overline{v}\,dx - ik\int_{\Gamma_1} u\overline{v}\,d\gamma(x). \tag{5.4}$$

We recall the following result which is a consequence of the Lax–Milgram theorem.

LEMMA 5.1. *For all $f \in \mathcal{V}'_\Omega$, there exists a unique $u_f \in \mathcal{V}_\Omega$ such that*

$$b(u_f, v) = \langle f, v\rangle_{\mathcal{V}'_\Omega,\mathcal{V}_\Omega}.$$

The operator $f \mapsto u_f$ is continuous from $\mathcal{V}'_\Omega$ to $\mathcal{V}_\Omega$.

We define

$$\begin{array}{cccc} \mathcal{C}: & \mathcal{V}_\Omega & \longrightarrow & \mathcal{V}_\Omega, \\ & u & \longmapsto & \mathcal{C}u \end{array}$$

such that

$$b(\mathcal{C}u, v) + c(u,v) = 0 \quad \forall v \in \mathcal{V}_\Omega. \tag{5.5}$$

We have the following lemma.

LEMMA 5.2. *The operator $\mathcal{C}$ is compact.*

Proof. By Lemma 5.1, it suffices to prove that the operator

$$u \longmapsto c(u,.)$$

from $\mathcal{V}_\Omega$ to $\mathcal{V}'_\Omega$ is compact. Let (u_i) be a sequence bounded in $\mathcal{V}_\Omega$. The imbeddings $\mathcal{V}_\Omega \to L^2(\Omega)$ and $H^{\frac{1}{2}}_{00}(\Gamma_1) \to L^2(\Gamma_1)$ are compact; then there exists a subsequence always denoted by (u_i) such that

$$u_i \to w_1 \text{ in } L^2(\Omega)$$

and

$$\gamma_0 u_i \to w_2 \text{ in } L^2(\Gamma_1).$$

Then

$$c(u_i,.) \to l^{w_2}_{w_1} \text{ in } \mathcal{V}'_\Omega,$$

where $l^{w_2}_{w_1}$ is defined by

$$\left\langle l^{w_2}_{w_1}, v\right\rangle_{\mathcal{V}'_\Omega,\mathcal{V}_\Omega} = -(1+k^2)\int_\Omega w_1\overline{v}\,dx - ik\int_{\Gamma_1} w_2\overline{v}\,d\gamma(x) \quad \forall v \in \mathcal{V}_\Omega.$$

Hence the operator $\mathcal{C}$ is compact. ◻

Using (5.5), problem (5.1) can be written as follows: find $u \in \mathcal{V}_\Omega$ such that

$$b((I-\mathcal{C})u, v) = l(v) \quad \forall v \in \mathcal{V}_\Omega. \tag{5.6}$$

We have the following lemma.

Lemma 5.3. *For $k \in \{k \in \mathbb{C}^*/\Im(k) \geq 0\}$, the following problem has no nontrivial solution: find $u \in \mathcal{V}_\Omega$ such that*

$$a(u,v) = 0 \quad \forall v \in \mathcal{V}_\Omega. \tag{5.7}$$

Proof. Let u be a solution to problem (5.7). For $v = u$, we have

$$a(u,u) = 0.$$

Then

$$\int_\Omega |\nabla u|^2\, dx - k^2 \int_\Omega |u|^2\, dx - ik \int_{\Gamma_1} |u|^2\, d\gamma(x) = 0. \tag{5.8}$$

By writing $k = k_1 + ik_2$, where $(k_1, k_2) \in \mathbb{R}^2$ and using (5.8), we obtain

$$\int_\Omega |\nabla u|^2\, dx - (k_1^2 - k_2^2) \int_\Omega |u|^2\, dx + k_2 \int_{\Gamma_1} |u|^2\, d\gamma(x) = 0 \tag{5.9}$$

and

$$k_1 \int_{\Gamma_1} |u|^2\, d\gamma(x) + 2k_1k_2 \int_\Omega |u|^2\, dx = 0. \tag{5.10}$$

Two cases can arise:

- First case: $k_2 > 0$. If $k_1 = 0$, using (5.9) we obtain

  $$\int_\Omega |\nabla u|^2\, dx + k_2^2 \int_\Omega |u|^2\, dx + k_2 \int_{\Gamma_1} |u|^2\, d\gamma(x) = 0.$$

  Then $u = 0$ in Ω. If $k_1 \neq 0$, using (5.10) we obtain

  $$\int_{\Gamma_1} |u|^2\, d\gamma(x) + 2k_2 \int_\Omega |u|^2\, dx = 0.$$

  Then $u = 0$ in Ω.
- Second case: $k_2 = 0$ and $k_1 \neq 0$. Using (5.10), we obtain

  $$u = 0 \quad \text{on } \Gamma_1.$$

  Let $\tilde{\Omega}$ be a regular domain containing Ω and so that $\Gamma_0 \subset \partial\tilde{\Omega}$. Extending u by zero in $\tilde{\Omega} \setminus \Omega$, we obtain a function $\tilde{u}$ that satisfies

  $$\Delta\tilde{u} + k^2\tilde{u} = 0 \quad \text{in } \mathcal{D}'(\tilde{\Omega}).$$

  This extension is analytic; it is equal to zero in an open subset of a connected domain; thus $\tilde{u} = 0$ in $\tilde{\Omega}$.

This completes the proof. $\square$

By Lemmas 5.2 and 5.3, and by using the Fredholm alternative, we obtain the following result.

THEOREM 5.4. *For* $k \in \{k \in \mathbb{C}^*/\Im(k) \geq 0\}$*, problem* (5.1) *has one and only one solution.*

5.2. The inf-sup condition. Our aim is to prove that the sesquilinear form a_0 defined by (3.9) for $\varepsilon = 0$ satisfies the inf-sup condition (see Hypothesis 2). We have the following lemma.

LEMMA 5.5. *The sesquilinear form* a *defined in* (5.1) *satisfies the* inf-sup *condition.*

Proof. Let u $\in \mathcal{V}_\Omega$. We set $v = (I - \mathcal{C})u$, where $\mathcal{C}$ is the operator defined by (5.5). According to (5.5), we have

$$\begin{aligned} a(u, v) &= b(v, v) \\ &= \|(I - \mathcal{C})u\|_{\mathcal{V}_\Omega} \|v\|_{\mathcal{V}_\Omega} \\ &\geq \alpha \|u\|_{\mathcal{V}_\Omega} \|v\|_{\mathcal{V}_\Omega}, \end{aligned}$$

where $\alpha = \|(I - \mathcal{C})^{-1}\|^{-1}_{\mathcal{L}(\mathcal{V}_\Omega, \mathcal{V}_\Omega)}$. Thus the sesquilinear form a satisfies the inf-sup condition. □

We have the following result.

PROPOSITION 5.6. *The sesquilinear form* a_0 *satisfies the* inf-sup *condition.*

Proof. We have

$$a_0(u, v) = \int_{\Omega_R} (\nabla u.\overline{\nabla v} - k^2 u\overline{v})\, dx + \int_{\Sigma_R} (T^0 u)\overline{v}\, d\gamma(x) - ik \int_{\Gamma_1} u\overline{v}\, d\gamma(x) \quad \forall u, v \in \mathcal{V}_R.$$

For all $u \in \mathcal{V}_R$ we set

$$\tilde{u} = \begin{cases} u \text{ in } \Omega_R, \\ u_\psi^0 \text{ in } B(x, R), \end{cases}$$

where $\psi = u_{|\Sigma_R}$ and u_ψ^0 is the solution to

$$\begin{cases} \Delta u_\psi^0 + k^2 u_\psi^0 = 0 & \text{in } B(x, R), \\ u_\psi^0 = \psi & \text{on } \Sigma_R. \end{cases}$$

It can easily be proved that

$$a_0(u, v_{|\Omega_R}) = a(\tilde{u}, v) \quad \forall u \in \mathcal{V}_R, \quad \forall v \in \mathcal{V}_\Omega.$$

According to Lemma 5.5, there exists $v \in \mathcal{V}_\Omega, v \neq 0$, such that

$$\begin{aligned} a_0(u, v_{|\Omega_R}) = a(\tilde{u}, v) &\geq \alpha \|\tilde{u}\|_{\mathcal{V}_\Omega} \|v\|_{\mathcal{V}_\Omega} \\ &\geq \alpha \|u\|_{\mathcal{V}_R} \|v_{|\Omega_R}\|_{\mathcal{V}_R}. \end{aligned}$$

This completes the proof. □

5.3. Some useful inequalities. We have the following proposition.

PROPOSITION 5.7. *There exists* $c > 0$ *such that*

$$\left| \varepsilon \frac{J_n(kR)}{J_n(k\varepsilon)} \right| \geq c \left(\frac{R}{\varepsilon} \right)^{n-1} \quad \forall n \geq n_0, \quad \forall \varepsilon < \varepsilon_0.$$

Proof. The Bessel function $J_n(z)$ is defined by

$$J_n(z) = \left(\frac{1}{2}z\right)^n \sum_{p=0}^{+\infty} \frac{(-\frac{1}{4}z^2)^p}{p!\Gamma(n+p+1)}.$$

Then we have

$$\begin{aligned}
\varepsilon\frac{J_n(kR)}{J_n(k\varepsilon)} &= \varepsilon\left(\frac{R}{\varepsilon}\right)^n \frac{\displaystyle\sum_{p=0}^{+\infty} \frac{(-\frac{1}{4}k^2R^2)^p}{p!\Gamma(n+p+1)}}{\displaystyle\sum_{p=0}^{+\infty} \frac{(-\frac{1}{4}k^2\varepsilon^2)^p}{p!\Gamma(n+p+1)}} \\
&= \varepsilon\left(\frac{R}{\varepsilon}\right)^n \frac{(\Gamma(n+1))^{-1} + \displaystyle\sum_{p=1}^{+\infty} \frac{(-\frac{1}{4}k^2R^2)^p}{p!\Gamma(n+p+1)}}{(\Gamma(n+1))^{-1} + \displaystyle\sum_{p=1}^{+\infty} \frac{(-\frac{1}{4}k^2\varepsilon^2)^p}{p!\Gamma(n+p+1)}} \\
&= \varepsilon\left(\frac{R}{\varepsilon}\right)^n \frac{1 + \displaystyle\sum_{p=1}^{+\infty} \frac{n!}{p!(n+p)!}\left(-\frac{1}{4}k^2R^2\right)^p}{1 + \displaystyle\sum_{p=1}^{+\infty} \frac{n!}{p!(n+p)!}\left(-\frac{1}{4}k^2\varepsilon^2\right)^p} \\
&= \left(\frac{R}{\varepsilon}\right)^{n-1} u_n(\varepsilon),
\end{aligned}$$

where $u_n(\varepsilon)$ is defined by

$$u_n(\varepsilon) = \frac{R + \displaystyle\sum_{p=1}^{+\infty} \frac{Rn!}{p!(n+p)!}\left(-\frac{1}{4}k^2R^2\right)^p}{1 + \displaystyle\sum_{p=1}^{+\infty} \frac{n!}{p!(n+p)!}\left(-\frac{1}{4}k^2\varepsilon^2\right)^p}.$$

It is easy to see that the series which intervene in the expression of $u_n(\varepsilon)$ converge normally with respect to (n, ε). Hence, we have

$$\lim_{(n,\varepsilon)\to(\infty,0)} u_n(\varepsilon) = R.$$

Using the limit definition, there exists $c > 0$ such that

$$|u_n(\varepsilon)| \geq c \quad \forall n \geq n_0, \quad \forall \varepsilon < \varepsilon_0.$$

This completes the proof. □

By the same techniques we obtain the following result.

Proposition 5.8. *There exists $c > 0$ such that*

$$\left|\frac{Y_n(k\varepsilon)}{Y_n(kR)}\right| \geq c\left(\frac{R}{\varepsilon}\right)^n \qquad \forall n \geq n_0, \quad \forall \varepsilon < \varepsilon_0.$$

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