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# OPTIMALITY CONDITIONS FOR SHAPE AND TOPOLOGY OPTIMIZATION SUBJECT TO A CONE CONSTRAINT*

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**Abstract.** This paper provides first order necessary optimality conditions for simultaneous shape and topology optimization subject to a cone constraint. These conditions are expressed with the help of the shape and topological derivatives of the objective and constraint functionals. Several examples of applications are given.

**Key words.** shape optimization, topology optimization, shape derivative, topological derivative, optimality condition, Lagrange multiplier

**AMS subject classifications.** 49K20, 49Q10, 35J05

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## 1. Introduction.

The classical theory of shape optimization consists of analyzing the behavior of a shape functional with respect to a small deformation of the domain, where each point moves along a direction represented by a displacement field [20, 24, 30]. In the formulation of [20, 24], the shape derivative is defined as a Fréchet derivative, which allows us to recast, at least locally, shape optimization problems as differentiable optimization problems. Therefore, optimality conditions can be derived straightforwardly by applying general results of nonlinear programming in Banach spaces. Unfortunately, this approach does not apply when one wants to allow topology variations. In fact, in this context, the set of attainable domains cannot be equipped with a structure of vector space in a natural and convenient way. In order to bypass this difficulty and still construct topology optimization algorithms, most authors use relaxation methods [2, 9, 10, 12, 15]. However, this point of view can hardly lead to optimality conditions for the nonrelaxed problem. Here, we follow another approach which relies on the notion of topological derivative. The principle is to analyze the sensitivity of the shape functional with respect to topology perturbations, typically the nucleation of small holes in the domain [4, 5, 16, 17, 19, 25, 28]. In the unconstrained situation, the nonnegativity of the topological derivative in the domain is an obvious necessary optimality condition, which forms the basis of several topology optimization algorithms [7, 13, 19, 26]. This topological optimality condition inside the domain can be complemented by the geometrical optimality condition at the boundary, namely, the vanishing of the shape derivative [29].

In this contribution, we investigate the constrained situation for simultaneous shape and topology perturbations. We establish first order necessary optimality conditions in the framework of a cone constraint of arbitrary dimension. The basic requirement is that the objective and constraint functionals admit suitable asymptotic expansions with respect to simultaneous shape and topology perturbations. We show that, through an appropriate formulation, the problem can be treated with the help of an adaptation of the classical nonlinear programming theory. More precisely, we prove that the convexity of some tangent set to the values taken by the aforementioned

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functionals is sufficient to guarantee the existence of Fritz John multipliers which, in turn, yields the existence of Lagrange multipliers under a Slater-type constraint qualification condition. To verify this convexity assumption, it turns out that a finite number of topology perturbations at arbitrary locations have to be considered at the same time, except in the special case of the volume constraint. This is a significant difference from the unconstrained case, where it was sufficient to consider a single perturbation. In practice, the topological derivative is generally additive with respect to the number of perturbations. But the proof of this statement requires a thorough analysis of the behavior of the direct and adjoint states, unless the topological derivative has been obtained by means of a truncation technique like in [19]. When the constraint is infinite dimensional, the asymptotic expansion of the constraint functional must also have a uniform remainder with respect to the constraint. We give a complete proof of the additivity and uniformity of the asymptotic expansion on an example of multiple loads, and we state the resulting necessary optimality conditions. The numerical solution of these optimality conditions is addressed in [6].

The paper is organized as follows. An abstract problem is first studied in section 2. It is applied in the setting of shape and topology optimization in section 3. In section 4 we provide a first example with a single constraint, based on asymptotic expansions derived in [29]. Section 5 is devoted to the example of multiple loads mentioned above. A third example, which involves a pointwise state constraint, is studied in section 6. In this case, the needed sensitivity formulas are known [18].

## 2. An abstract result.

As stated in the introduction, the standard theory of mathematical programming does not apply directly to topological shape optimization problems. The purpose of this section is to propose a natural adaptation of the classical framework (see, e.g., [8, 11]) to such problems. We provide an abstract presentation for clarity rather than generality.

### 2.1. Problem statement.

Let $\mathscr{E}$ be an arbitrary set, $Y$ be a separated and locally convex topological vector space, and $K$ be a closed convex cone of $Y$ with nonempty interior. We consider two functionals

$$J = \begin{cases} \mathscr{E} \to \mathbb{R}, \\ \Omega \mapsto J(\Omega), \end{cases} \qquad G = \begin{cases} \mathscr{E} \to Y, \\ \Omega \mapsto G(\Omega). \end{cases}$$

We are seeking optimality conditions associated with the minimization problem

$$(\mathscr{P}) \qquad \underset{\Omega \in \mathscr{E}}{\text{minimize}}\, J(\Omega) \quad \text{subject to } G(\Omega) \in -K.$$

### 2.2. Perturbations.

Let $\Omega \in \mathscr{E}$ be given. We call *perturbation of* $\Omega$ any map $\xi : [0, \tau_\xi] \to \mathscr{E}$, with $\tau_\xi > 0$, such that $\xi(0) = \Omega$. If $\xi(t) = \Omega$ for all $t \in [0, \tau_\xi]$, we say that $\xi$ is a *zero perturbation*.

Let $Per(\Omega)$ be a set of perturbations of $\Omega$. We assume that $Per(\Omega)$ contains at least a zero perturbation, which we denote by 0. We say that $\Omega$ is $Per(\Omega)$-*optimal* if $G(\Omega) \in -K$ and

$$\forall \xi \in Per(\Omega),\ \exists \tau \in (0, \tau_\xi]\ \forall t \in [0, \tau],\ G(\xi(t)) \in -K \Rightarrow J(\xi(t)) \geq J(\Omega).$$

### 2.3. Directional derivative.

Let $\xi$ be a perturbation of $\Omega$. We say that a function $F : \mathscr{E} \to Z$, where $Z$ is a separated topological vector space, is *differentiable in* $\Omega$ *in the direction* $\xi$ if $F \circ \xi$ is differentiable at zero. Then we define the *derivative of* $F$ *in* $\Omega$ *in the direction* $\xi$ by

$$F'(\Omega, \xi) = (F \circ \xi)'(0).$$

In particular, if $F$ is differentiable in $\Omega$ in the direction $\xi$, there holds

$$F(\xi(t)) = F(\Omega) + tF'(\Omega,\xi) + o(t) \qquad \forall t \in [0,\tau_\xi],$$

with $\lim_{t\to 0}\frac{o(t)}{t} = 0$. We say that $F$ is $Per(\Omega)$-*differentiable* if $F$ is differentiable in every direction $\xi \in Per(\Omega)$.

**2.4. Existence of Fritz John multipliers.** We assume that the functionals $J$ and $G$ are $Per(\Omega)$-differentiable. Throughout, the following standard notation will be used: $Y'$ stands for the topological dual of $Y$, $\langle .,. \rangle$ denotes the duality pairing between $Y'$ and $Y$, and $K^+$ is the positive polar cone of $K$.

LEMMA 2.1. *If $\Omega$ is $Per(\Omega)$-optimal, then for all $\xi \in Per(\Omega)$,*

$$G(\Omega) + G'(\Omega,\xi) \in \mathrm{Int}(-K) \Rightarrow J'(\Omega,\xi) \geq 0.$$

*Proof.* Consider a perturbation $\xi \in Per(\Omega)$ such that $G(\Omega)+G'(\Omega,\xi) \in \mathrm{Int}(-K)$. By definition of the directional derivatives, we have for all $t \in [0,\tau_\xi]$

$$J(\xi(t)) = J(\Omega) + tJ'(\Omega,\xi) + o(t),$$
$$G(\xi(t)) = G(\Omega) + tG'(\Omega,\xi) + o(t).$$

Thus, for $t$ small enough, $G(\xi(t)) = (1-t)G(\Omega) + t\,[G(\Omega) + G'(\Omega,\xi) + o(1)] \in -K$. It follows that $J(\xi(t)) - J(\Omega) = tJ'(\Omega,\xi) + o(t) \geq 0$. Dividing by $t > 0$ and letting $t$ go to zero yields $J'(\Omega,\xi) \geq 0$. ∎

In order to apply a separation argument, similarly to [8], we define the set

$$\mathscr{T}(J,G,\Omega,Per(\Omega)) = \{(J'(\Omega,\xi)+\alpha, G(\Omega)+G'(\Omega,\xi)+y), \alpha > 0, y \in \mathrm{Int}(K), \xi \in Per(\Omega)\}. \tag{2.1}$$

LEMMA 2.2. *If $\Omega$ is $Per(\Omega)$-optimal and $\mathscr{T}(J,G,\Omega,Per(\Omega))$ is convex, then there exists a pair of Fritz John multipliers $(\gamma,y^*) \in (\mathbb{R}^+ \times K^+) \setminus \{(0,0)\}$ such that*

$$\gamma J'(\Omega,\xi) + \langle y^*, G'(\Omega,\xi)\rangle \geq 0 \qquad \forall \xi \in Per(\Omega), \tag{2.2}$$

$$\langle y^*, G(\Omega)\rangle = 0. \tag{2.3}$$

*Proof.* Clearly, the set $\mathscr{T}(J,G,\Omega,Per(\Omega))$ is open and nonempty. Assume that $(0,0) \in \mathscr{T}(J,G,\Omega,Per(\Omega))$. Then there exist $\xi \in Per(\Omega)$, $\alpha > 0$, and $y \in \mathrm{Int}(K)$ such that

$$(0,0) = (J'(\Omega,\xi)+\alpha, G(\Omega)+G'(\Omega,\xi)+y).$$

Hence $G(\Omega)+G'(\Omega,\xi) = -y \in \mathrm{Int}(-K)$ and $J'(\Omega,\xi) = -\alpha < 0$, which is impossible by virtue of Lemma 2.1. Therefore, by a geometric form of the Hahn–Banach theorem, there exists $(\gamma,y^*) \in \mathbb{R} \times Y' \setminus \{(0,0)\}$ such that

$$\gamma s + \langle y^*, z\rangle \geq 0 \qquad \forall (s,z) \in \mathscr{T}(J,G,\Omega,Per(\Omega)).$$

In other words,

$$\gamma(J'(\Omega,\xi)+\alpha) + \langle y^*, G(\Omega)+G'(\Omega,\xi)+y\rangle \geq 0 \qquad \forall (\xi,\alpha,y) \in Per(\Omega)\times\mathbb{R}_+^*\times\mathrm{Int}(K).$$

As $K$ is closed and convex with nonempty interior, we have $\mathrm{cl}(\mathrm{Int}\,K) = K$. Hence, by continuity, the above inequality holds true for all $(\xi,\alpha,y) \in Per(\Omega)\times\mathbb{R}_+\times K$.

Taking $\xi = 0$, $\alpha = 0$, and $y = 0$ entails $\langle y^*, G(\Omega)\rangle \geq 0$. Taking $\xi = 0$, $\alpha = 0$, and $y = -2G(\Omega)$ entails $\langle y^*, G(\Omega)\rangle \leq 0$. Thus $\langle y^*, G(\Omega)\rangle = 0$ and

$$\gamma(J'(\Omega,\xi) + \alpha) + \langle y^*, G'(\Omega,\xi) + y\rangle \geq 0 \qquad \forall(\xi,\alpha,y) \in Per(\Omega) \times \mathbb{R}^+ \times K.$$

Fixing $\alpha = 0$ and $y = 0$ leads to (2.2). Fixing now $\xi = 0$ results in

$$\gamma\alpha + \langle y^*, y\rangle \geq 0 \qquad \forall(\alpha, y) \in \mathbb{R}_+ \times K.$$

Choosing successively $y = 0$ and $\alpha = 0$ results in $\gamma \geq 0$ and $y^* \in K^+$. ☐

**2.5. Existence of Lagrange multipliers.**

PROPOSITION 2.3. *Assume that $\Omega$ is $Per(\Omega)$-optimal, $\mathscr{T}(J, G, \Omega, Per(\Omega))$ is convex, and that the following constraint qualification holds: there exists $\bar{\xi} \in Per(\Omega)$ such that*

$$G(\Omega) + G'(\Omega, \bar{\xi}) \in \mathrm{Int}(-K).$$

*Then there exists a Lagrange multiplier $y^* \in K^+$ such that*

$$J'(\Omega,\xi) + \langle y^*, G'(\Omega,\xi)\rangle \geq 0 \qquad \forall\xi \in Per(\Omega), \tag{2.4}$$

$$\langle y^*, G(\Omega)\rangle = 0. \tag{2.5}$$

*Proof.* It suffices to prove that, in Lemma 2.2, $\gamma > 0$. Suppose that $\gamma = 0$. Then $y^* \neq 0$ and (2.2) implies

$$\langle y^*, G'(\Omega,\xi)\rangle \geq 0 \qquad \forall\xi \in Per(\Omega).$$

Due to (2.3), we can write

$$\langle y^*, G(\Omega) + G'(\Omega,\xi)\rangle \geq 0 \qquad \forall\xi \in Per(\Omega).$$

In particular, by setting $\bar{y} = G(\Omega) + G'(\Omega, \bar{\xi}) \in \mathrm{Int}(-K)$, we get $\langle y^*, \bar{y}\rangle \geq 0$. Since $y^* \neq 0$, there exists $\delta \in Y$ such that $\bar{y} + \delta \in -K$ and $\langle y^*, \delta\rangle > 0$. Therefore, $\langle y^*, \bar{y}\rangle = \langle y^*, \bar{y} + \delta\rangle - \langle y^*, \delta\rangle < 0$, which constitutes a contradiction. ☐

## 3. Application to shape and topology optimization.

We now assume that $\mathscr{E}$ is a set of domains (open and connected subsets) of $\mathbb{R}^d$, $d = 2$, or $d = 3$.

### 3.1. Shape and topology perturbations.

Starting from an arbitrary domain $\Omega \in \mathscr{E}$, we consider the following two types of perturbations:

- *Shape perturbations.* Following the classical approach introduced in [24], we represent a shape perturbation by a displacement field $V \in W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$. We define the perturbed domain

$$\Omega(V) = (I + V)(\Omega),$$

  with $I$ the identity mapping of $\mathbb{R}^d$.
- *Topology perturbations.* These consist of drilling holes in $\Omega$ as well as nucleating disconnected parts outside $\Omega$. As stated in the introduction, it is crucial for the analysis to allow an arbitrary number of such perturbations to be done at the same time. Therefore a topology perturbation is characterized by a

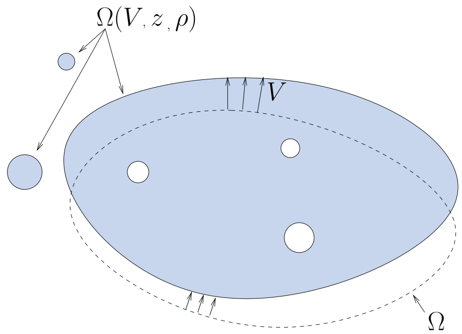


Fig. 1. *Domain perturbations.*

number $m \in \mathbb{N}^* := \mathbb{N} \setminus \{0\}$, a family of distinct points $z = (z_1, \ldots, z_m) \in (\mathbb{R}^d)^m$, and a family of radii $\rho = (\rho_1, \ldots, \rho_m) \in \mathbb{R}_+^m$. It leads to the domain

$$\Omega(z, \rho) = \left( \Omega \setminus \bigcup_{\substack{z_i \in \mathrm{Int}(\Omega) \\ i=1,\ldots,m}} \overline{\mathbb{B}(z_i, \rho_i)} \right) \cup \bigcup_{\substack{z_i \in \mathrm{Ext}(\Omega) \\ i=1,\ldots,m}} \mathbb{B}(z_i, \rho_i), \tag{3.1}$$

with $\mathbb{B}(z_i, \rho_i) = \{x \in \mathbb{R}^d \,|\, |x - z_i| < \rho_i\}$, and $\mathrm{Int}(\Omega)$ and $\mathrm{Ext}(\Omega)$ are the interior and the exterior of $\Omega$, respectively. Note that the points $z_i \in \partial\Omega$ are not taken into account. We point out that balls have been chosen merely to fix ideas. We denote by

$$\Omega(V, z, \rho) = \Omega(V)(z, \rho) = \left( (I + V)(\Omega) \setminus \bigcup_{\substack{z_i \in \mathrm{Int}(\Omega) \\ i=1,\ldots,m}} \overline{\mathbb{B}(z_i, \rho_i)} \right) \cup \bigcup_{\substack{z_i \in \mathrm{Ext}(\Omega) \\ i=1,\ldots,m}} \mathbb{B}(z_i, \rho_i)$$

the domain obtained after shape and topology perturbations (see Figure 1). In order to ensure that this new domain belongs to $\mathscr{E}$, we assume that there exists a convex cone $\mathcal{S}(\Omega) \subset W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$ of admissible directions of shape perturbations and a set $\mathcal{T}(\Omega) \subset \mathbb{R}^d \setminus \partial\Omega$ of admissible points for the topology perturbations satisfying the following property: for all $V \in \mathcal{S}(\Omega)$, $m \in \mathbb{N}^\star$, and $z \in \mathcal{T}(\Omega)^{[m]}$, there exist $t_0 > 0$ and $\rho_0 > 0$ such that

$$(t, \rho) \in [0, t_0] \times [0, \rho_0]^m \Rightarrow \Omega(tV, z, \rho) \in \mathscr{E}.$$

Here and subsequently, we use the notation

$$\mathcal{T}(\Omega)^{[m]} = \{z = (z_1, \ldots, z_m) \in \mathcal{T}(\Omega)^m, z_i \neq z_j \forall i \neq j\}.$$

To enter into the framework of section 2, we need to represent the magnitude of all perturbations by the single scalar parameter $t$. Thus we set

$$\xi(t; V, z, \beta) = \Omega(tV, z, h(t\beta)),$$

where $\beta = (\beta_1, \ldots, \beta_m) \in \mathbb{R}_+^m$, $h : \mathbb{R}_+ \to \mathbb{R}_+$ is a function such that $\lim_{s \to 0} h(s) = 0$, and $h(t\beta) = (h(t\beta_1), \ldots, h(t\beta_m))$. Then we define the set of perturbations

$$Per(\Omega) = \big\{\xi(.; V, z, \beta) : [0, \tau] \to \mathscr{E}, V \in \mathcal{S}(\Omega), z \in \mathcal{T}(\Omega)^{[m]}, \beta \in \mathbb{R}_+^m, m \in \mathbb{N}^*, \tau > 0\big\}. \tag{3.2}$$

Above, $\tau$ is chosen such that $t \leq \tau \Rightarrow \Omega(tV, z, h(t\beta)) \in \mathscr{E}$.

**3.2. Sensitivity of the functionals.** We assume that for all $V \in \mathcal{S}(\Omega)$, $m \in \mathbb{N}^*$, and $z \in \mathcal{T}(\Omega)^{[m]}$, the functionals $J$ and $G$ admit the asymptotic expansions

$$J(\Omega(tV, z, \rho)) = J(\Omega) + t\langle J'_S(\Omega), V\rangle + \sum_{i=1}^{m} f(\rho_i) J'_T(\Omega)(z_i) + o(t, f(\rho_1), \dots, f(\rho_m)), \tag{3.3}$$

$$G(\Omega(tV, z, \rho)) = G(\Omega) + tG'_S(\Omega)(V) + \sum_{i=1}^{m} f(\rho_i) G'_T(\Omega)(z_i) + o(t, f(\rho_1), \dots, f(\rho_m)). \tag{3.4}$$

Here, the function $f$ is a homeomorphism of $\mathbb{R}_+$ into itself (or, more generally, a homeomorphism of $[0, a]$ into $[0, b]$ for some $a, b > 0$) with $f(0) = 0$, and the remainder $o(t, f(\rho_1), \dots, f(\rho_m))$ satisfies

$$o(t, f(\rho_1), \dots, f(\rho_m)) = (t^2 + f(\rho_1)^2 + \cdots + f(\rho_m)^2)^{1/2} \varepsilon(t, f(\rho_1), \dots, f(\rho_m)),$$

with $\lim_{x \to 0} \varepsilon(x) = 0$. The functions $J'_S(\Omega) : \mathcal{S}(\Omega) \to \mathbb{R}$ and $G'_S(\Omega) : \mathcal{S}(\Omega) \to Y$ are linear. The functions $J'_T(\Omega) : \mathcal{T}(\Omega) \to \mathbb{R}$ and $G'_T(\Omega) : \mathcal{T}(\Omega) \to Y$ are arbitrary. We refer the reader to sections 4, 5, and 6 for the existence of such expansions in some practical situations. We emphasize that these expansions are known in many situations when a single perturbation is considered (i.e., a shape perturbation or a topology perturbation with $m = 1$). The linear functional $J'_S(\Omega)$ is the so-called shape derivative of $J$, and the function $J'_T(\Omega)$ is the corresponding topological derivative. The homeomorphism $f$ depends on the space dimension and, when the functionals involve the solution of a partial differential equation, on the boundary condition prescribed on $\partial\mathbb{B}(z_i, \rho_i)$ (see Table 1).

Setting $h = f^{-1}$, we derive straightforwardly that, for all $\beta \in \mathbb{R}_+^m$,

$$J(\xi(t; V, z, \beta)) = J(\Omega) + t\langle J'_S(\Omega), V\rangle + t\sum_{i=1}^{m} \beta_i J'_T(\Omega)(z_i) + o(t), \tag{3.5}$$

$$G(\xi(t; V, z, \beta)) = G(\Omega) + tG'_S(\Omega)(V) + t\sum_{i=1}^{m} \beta_i G'_T(\Omega)(z_i) + o(t). \tag{3.6}$$

Hence the functionals $J$ and $G$ are differentiable in the direction $\xi(.; V, z, \beta)$ and

$$J'(\Omega, \xi(.; V, z, \beta)) = \langle J'_S(\Omega), V\rangle + \sum_{i=1}^{m} \beta_i J'_T(\Omega)(z_i), \tag{3.7}$$

$$G'(\Omega, \xi(.; V, z, \beta)) = G'_S(\Omega)(V) + \sum_{i=1}^{m} \beta_i G'_T(\Omega)(z_i). \tag{3.8}$$

**3.3. Convexity of $\mathscr{T}(J, G, \Omega, Per(\Omega))$.**

LEMMA 3.1. *When $Per(\Omega)$ is defined by* (3.2) *and the directional derivatives are defined by* (3.7) *and* (3.8), *then the set $\mathscr{T}(J, G, \Omega, Per(\Omega))$ defined by* (2.1) *is convex.*

Table 1
*Expression of $f(\rho)$ for state dependent functionals.*

| Boundary condition | 2D | 3D |
|---|---|---|
| Dirichlet | $\frac{-1}{\ln \rho}$ | $\rho$ |
| Neumann or transmission | $\pi\rho^2$ | $\frac{4}{3}\pi\rho^3$ |

*Proof.* We have

$$\begin{aligned}&\mathscr{T}(J,G,\Omega,Per(\Omega))\\&\quad=\Bigg\{\Bigg(\langle J_S'(\Omega),V\rangle+\sum_{i=1}^m\beta_iJ_T'(\Omega)(z_i)+\alpha,G_S'(\Omega)(V)+\sum_{i=1}^m\beta_iG_T'(\Omega)(z_i)+y\Bigg),\\&V\in\mathcal{S}(\Omega),m\in\mathbb{N}^*,(z_1,\dots,z_m)\in\mathcal{T}(\Omega)^{[m]},(\beta_1,\dots,\beta_m)\in\mathbb{R}_+^m,\alpha>0,y\in\text{Int}(K)\Bigg\}.\end{aligned}$$

Consider two pairs $(a,b),(a',b')\in\mathscr{T}(J,G,\Omega,Per(\Omega))$. There exist $V\in\mathcal{S}(\Omega)$, $m\in\mathbb{N}^*$, $z=(z_1,\dots,z_m)\in\mathcal{T}(\Omega)^{[m]}$, $\beta=(\beta_1,\dots,\beta_m)\in\mathbb{R}_+^m$, $\alpha>0$, and $y\in\text{Int}(K)$ such that

$$a=\langle J_S'(\Omega),V\rangle+\sum_{i=1}^m\beta_iJ_T'(\Omega)(z_i)+\alpha,\qquad b=G_S'(\Omega)(V)+\sum_{i=1}^m\beta_iG_T'(\Omega)(z_i)+y.$$

Similarly, we write

$$a'=\langle J_S'(\Omega),V'\rangle+\sum_{j=1}^{m'}\beta_j'J_T'(\Omega)(z_j')+\alpha',\qquad b'=G_S'(\Omega)(V')+\sum_{j=1}^{m'}\beta_j'G_T'(\Omega)(z_j')+y'.$$

We define the sets $Z=\{z_i,i=1,\dots,m\}$, $Z'=\{z_j',j=1,\dots,m'\}$, $\mathcal{I}=\{i\in[\![1,m]\!],z_i\notin Z'\}$, $\mathcal{J}=\{j\in[\![1,m']\!],z_j'\notin Z\}$, $\mathcal{K}=\{(i,j)\in[\![1,m]\!]\times[\![1,m']\!],z_i=z_j'\}$. For any $\theta\in[0,1]$, we have

$$\begin{aligned}\theta a+(1-\theta)a'&=\langle J_S'(\Omega),\theta V+(1-\theta)V'\rangle+\sum_{i\in\mathcal{I}}\theta\beta_iJ_T'(\Omega)(z_i)+\sum_{j\in\mathcal{J}}(1-\theta)\beta_jJ_T'(\Omega)(z_j')\\&+\sum_{(i,j)\in\mathcal{K}}(\theta\beta_i+(1-\theta)\beta_j')J_T'(\Omega)(z_i)+\theta\alpha+(1-\theta)\alpha'.\end{aligned}$$

Obviously, we have an analogous expression for $\theta b+(1-\theta)b'$. We deduce that $(\theta a+(1-\theta)a',\theta b+(1-\theta)b')\in\mathscr{T}(J,G,\Omega,Per(\Omega))$. ∎

*Remark* 1. In the special case of the volume constraint $G(\Omega)=|\Omega|-M$, $M>0$, it turns out that the convexity of $\mathscr{T}(J,G,\Omega,Per(\Omega))$ can be equally obtained by considering a single topology perturbation ($m=1$), provided that $J_T'(\Omega)$ is continuous over $\Omega$. Assume for simplicity that $\mathcal{S}(\Omega)=\emptyset$ and $\mathcal{T}(\Omega)=\Omega$. Then

$$\mathscr{T}(J,G,\Omega,Per(\Omega))=\big\{\big(\beta J_T'(\Omega)(z)+\alpha,\beta G_T'(\Omega)(z)+y\big),z\in\Omega,\beta\geq 0,\alpha>0,y>0\big\}.$$

Yet $G_T'(\Omega)(z)=-1$ for all $z\in\Omega$, which yields $\mathscr{T}(J,G,\Omega,Per(\Omega))=\mathbb{R}_+(J_T'(\Omega)(\Omega)\times\{-1\})+\mathbb{R}_+^*\times\mathbb{R}_+^*$. Since $J_T'(\Omega)(\Omega)$ is a real interval (because $\Omega$ is connected), we conclude that $\mathscr{T}(J,G,\Omega,Per(\Omega))$ is convex.

**3.4. Optimality conditions.** We are now in position to state our main result.

THEOREM 3.2 (necessary optimality conditions). *Let $\Omega$ be a $Per(\Omega)$-optimal domain, where $Per(\Omega)$ is defined by* (3.2). *We assume that the functionals $J$ and $G$ admit shape and topological sensitivities of the form* (3.3) *and* (3.4). *Moreover, we suppose that the following constraint qualification condition is fulfilled: there exists $V \in \mathcal{S}(\Omega)$, $m \in \mathbb{N}^*$, $(z_1, \ldots, z_m) \in \mathcal{T}(\Omega)^{[m]}$, $(\beta_1, \ldots, \beta_m) \in \mathbb{R}_+^m$ such that*

$$\text{(CQ)} \qquad G(\Omega) + G'_S(\Omega)(V) + \sum_{i=1}^{m} \beta_i G'_T(\Omega)(z_i) \in \operatorname{Int}(-K).$$

*Then there exists $\mu \in K^+$ such that*

$$\text{(3.9)} \qquad \langle \mu, G(\Omega) \rangle = 0,$$

$$\text{(3.10)} \qquad \forall V \in \mathcal{S}(\Omega), \quad \langle J'_S(\Omega) + \mu \circ G'_S(\Omega), V \rangle \geq 0,$$

$$\text{(3.11)} \qquad \forall x \in \mathcal{T}(\Omega), \quad J'_T(\Omega)(x) + \langle \mu, G'_T(\Omega)(x) \rangle \geq 0.$$

*Proof.* In view of Lemma 3.1, we can apply Proposition 2.3 with the notation and assumptions of this section. Therefore there exists $\mu \in K^+$ such that $\langle \mu, G(\Omega) \rangle = 0$ and, for all $V \in \mathcal{S}(\Omega), m \in \mathbb{N}^*, z \in \mathcal{T}(\Omega)^{[m]}$, and $\beta \in \mathbb{R}_+^m$,

$$\langle J'_S(\Omega), V \rangle + \sum_{i=1}^{m} \beta_i J'_T(\Omega)(z_i) + \left\langle \mu, G'_S(\Omega)(V) + \sum_{i=1}^{m} \beta_i G'_T(\Omega)(z_i) \right\rangle \geq 0.$$

Choosing $\beta = 0$ yields (3.10). Choosing $V = 0$, $m = 1$, and $\beta_1 = 1$ provides (3.11). ∎

*Remark* 2. The condition (CQ) in Theorem 3.2 can be seen as a translation of the linearized Slater constraint qualification, which is equivalent to Robinson's constraint qualification when $K$ has nonempty interior [11]. This condition is very general and is widely used in control theory [21]. The assumption $\operatorname{Int}(K) \neq \emptyset$ proves to be verified in many cases of interest. It is, for instance, commonly admitted in problems involving state constraints of inequality type.

**4. First example.** In this section we revisit an example proposed in [29]. Let $\Omega$ be the two-dimensional ball $\mathbb{B}(0, R)$ for some $R \in (0, 1]$ to be determined. We denote by $\Gamma$ its boundary. We consider the boundary value problem

$$\begin{cases} -\Delta u_\Omega = 1 & \text{in} \quad \Omega, \\ u_\Omega = 0 & \text{on} \quad \Gamma. \end{cases}$$

We take $\mathscr{E}$ as the set of all domains of $\mathbb{R}^2$ and take $\mathcal{S}(\Omega)$ as the set of all maps $V \in \mathcal{C}^4(\mathbb{R}^2, \mathbb{R}^2)$ such that $\operatorname{supp}(V) \subset \overline{\mathbb{B}(0, R_0)}$, with $R_0 > R$. In a first step, we do not consider any topology perturbation, i.e., $\mathcal{T}(\Omega) = \emptyset$. We consider the objective functional

$$J(\Omega) = -\int_\Omega u_\Omega^2 dx$$

to be minimized, and the constraint functional

$$G(\Omega) = |\Omega| - \pi$$

to be nonpositive. Thus the relevant sets are $Y = Y' = \mathbb{R}$, $K = K^+ = \mathbb{R}_+$. According to [29], the asymptotic expansions (3.3) and (3.4) hold true with

$$\langle J'_S(\Omega), V\rangle = -\int_\Gamma \partial_n u_\Omega \partial_n v_\Omega (V.n) ds,$$
$$G'_S(\Omega)(V) = \int_\Gamma V.n ds.$$

Above, $n$ is the outward unit normal to $\Gamma$ and $v_\Omega$ is the adjoint state solution of

$$\begin{cases} -\Delta v_\Omega &= 2u_\Omega \quad \text{in} \quad \Omega, \\ v_\Omega &= 0 \quad \text{on} \quad \Gamma. \end{cases}$$

Here, the constraint qualification condition (CQ) is guaranteed regardless of $\Omega$ (take, for instance, $V(x) = -\eta(|x|)x$, where $\eta$ is a smooth function such that $\eta(s) = 1$ if $s \leq R$ and $\eta(s) = 0$ if $s \geq R_0$). Hence the necessary optimality conditions read as follows: there exists $\mu \in \mathbb{R}_+$ such that

$$\mu(|\Omega| - \pi) = 0, \tag{4.1}$$

$$\forall V \in \mathcal{S}(\Omega), \qquad \int_\Gamma [\mu - \partial_n u_\Omega \partial_n v_\Omega] (V.n) ds \geq 0. \tag{4.2}$$

By linearity of the space $\mathcal{S}(\Omega)$, (4.2) is equivalent to

$$\mu - \partial_n u_\Omega \partial_n v_\Omega = 0 \quad \text{a.e. on } \Gamma. \tag{4.3}$$

In this simple example, the direct and adjoint states can be calculated explicitly. We have in polar coordinates with $r = |x|$

$$u_\Omega(x) = \frac{1}{4}(R^2 - r^2),$$
$$v_\Omega(x) = \frac{1}{32}(r^4 - R^4) - \frac{1}{8}R^2(r^2 - R^2).$$

Plugging these expressions into (4.3) yields

$$\mu - \frac{1}{16}R^4 = 0. \tag{4.4}$$

Since $R > 0$, we have $\mu > 0$. Hence $R = 1$ by virtue of (4.1). From (4.4) we obtain that $\mu = 1/16$. Thus the first-order shape optimality conditions are fulfilled for the domain $\Omega = \mathbb{B}(0, 1)$. Let us now consider topology perturbations consisting of Neumann circular perforations whose centers can be located at every point of $\mathcal{T}(\Omega) = \Omega$. It is proved in [29] that the expansions (3.3) and (3.4) hold with the topological contributions given by

$$f(\rho) = \pi\rho^2,$$
$$J'_T(\Omega)(z) = u_\Omega(z)^2 + v_\Omega(z) - 2\nabla u_\Omega(z).\nabla v_\Omega(z),$$
$$G'_T(\Omega)(z) = -1.$$

Hence the topological optimality condition reads as

$$-\mu + \left[u_\Omega(z)^2 + v_\Omega(z) - 2\nabla u_\Omega(z).\nabla v_\Omega(z)\right] \geq 0 \quad \forall z \in \Omega.$$

Yet the function

$$r \mapsto \frac{1}{16} - \left[u_\Omega(r)^2 + v_\Omega(r) - 2\nabla u_\Omega(r).\nabla v_\Omega(r)\right],$$

whose graph is shown in [29], is positive when $r > r_0 \approx 0.45$. We conclude that the unit ball is no more optimal when topology perturbations are allowed. We retrieve the conclusion obtained in [29], without duality argument, by considering perturbations at a constant area.

## 5. Second example.

This example deals with shape and topology optimization with multiple loads. The constraint is an upper bound prescribed on the compliance associated with each load. When the loads are concentrated around a reference load, one speaks of robust compliance [14]. In comparison with the previous example, this problem has the following features:

- The space of constraints $Y$ is infinite dimensional;
- the exterior of $\Omega$ is occupied by some background phase, which enables the nucleation of material islands.

### 5.1. Problem statement.

Let $D$ be a bounded domain of $\mathbb{R}^2$ with a Lipschitz boundary made of two disjoint parts $\Gamma_D$ and $\Gamma_N$, and let $\mathscr{E}$ be the set of all subdomains of $D$. We assume that $\Gamma_D$ is of nonzero measure, and we define the function space $\mathcal{V} = \{u \in H^1(D), u_{|\Gamma_D} = 0\}$. We denote by $\mathcal{U}$ the topological dual of the space of traces on $\Gamma_N$ of functions of $\mathcal{V}$ for the $H^{1/2}(\Gamma_N)$ norm. Let $I$ be a compact subset of $\mathbb{R}$, and let $F : I \to \mathcal{U}$ be a continuous mapping. For each parameter $\eta \in I$ and each domain $\Omega \subset D$ we consider the boundary value problem

$$\left\{ \begin{array}{rcll} -\mathrm{div}\,(\alpha_\Omega \nabla u_\Omega(\eta)) & = & 0 & \text{in} \quad D, \\ \alpha_\Omega \nabla u_\Omega(\eta).n & = & F(\eta) & \text{on} \quad \Gamma_N, \\ u_\Omega(\eta) & = & 0 & \text{on} \quad \Gamma_D, \end{array} \right. \tag{5.1}$$

with

$$\alpha_\Omega = \left\{ \begin{array}{l} \alpha^+ \text{ in } \Omega, \\ \alpha^- \text{ in } D \setminus \Omega. \end{array} \right.$$

Above, $\alpha^+$ and $\alpha^-$ are two different positive constants and $n$ stands for the outward unit normal to $\Gamma_N$. The variational formulation of (5.1) reads as

$$\int_D \alpha_\Omega \nabla u_\Omega(\eta).\nabla v dx = \langle F(\eta), v\rangle \qquad \forall v \in \mathcal{V}. \tag{5.2}$$

We set

$$Y = \mathcal{C}(I),$$

the space of real-valued continuous functions on $I$, and

$$K = \{f \in Y, f(\eta) \geq 0\ \forall \eta \in I\}.$$

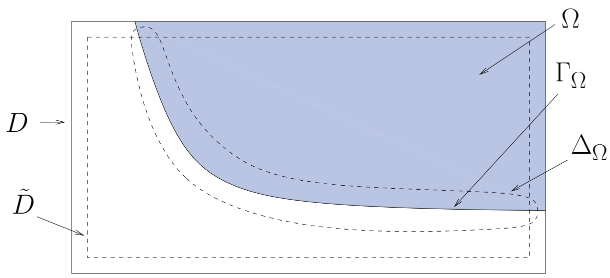


FIG. 2. *Geometrical configuration for the second example.*

We take as objective functional the area

$$J(\Omega) = |\Omega|,$$

and as constraint functional the energy (or compliance) translated by a given element $M$ of $Y$

$$G(\Omega) : \eta \mapsto \int_D \alpha_\Omega |\nabla u_\Omega(\eta)|^2 dx - M(\eta) = \langle F(\eta), u_\Omega(\eta)\rangle - M(\eta).$$

**5.2. Shape and topological sensitivities.**

**5.2.1. Domain perturbations.** Let $\Omega$ be a domain of $\mathcal{E}$. We assume that the interface $\Gamma_\Omega = \partial\Omega \cap D$ is of class $\mathcal{C}^{1,1}$. We consider simultaneous shape and topology perturbations of $\Omega$ as described in section 3, with the following sets:

$$\mathcal{S}(\Omega) = \{V \in \mathcal{C}^{1,1}(\mathbb{R}^2, \mathbb{R}^2), \text{supp}(V) \subset \tilde{D}\}, \qquad \tilde{D} \subset\subset D,$$

$$\mathcal{T}(\Omega) = D \setminus \Gamma_\Omega.$$

The displacement field $V \in \mathcal{S}(\Omega)$ and the centers $(z_1, \dots, z_m) \in \mathcal{T}(\Omega)^{[m]}$ being fixed, we choose $t_0 > 0$ and $\rho_0 > 0$ so that, for all $(t, \rho) \in [0, t_0] \times [0, \rho_0]^m$, the inclusions $\mathbb{B}(z_i, t\rho_i)$ are separated from each other, do not touch the boundary of $D$, and meet neither $\Gamma_\Omega$ nor a tubular neighborhood $\Delta_\Omega \subset\subset D$ of $\Gamma_\Omega \cap \tilde{D}$ in which the interface $\Gamma_{\Omega(tV)}$ evolves (see Figure 2).

The shape and topological sensitivities of the objective functional are obvious. We subsequently focus on those of the constraint functional.

**5.2.2. Preliminary lemmas.**

LEMMA 5.1. *Consider two domains $\omega$ and $\omega'$ of $\mathbb{R}^d$ with $\omega' \subset \omega$, and suppose that $\alpha \in L^\infty(\omega)$. We define the space*

$$\mathcal{H} = \{u \in H^1(\omega), -\text{div}(\alpha\nabla u) = 0 \text{ in } \omega\},$$

*where the divergence operator is defined in the sense of distributions. We assume that there exists $p \in (2, +\infty]$ such that*

$$\mathcal{H}_{|\omega'} := \{u_{|\omega'}, u \in \mathcal{H}\} \subset W^{1,p}(\omega').$$

*Then there exists a constant $c > 0$ such that*

$$\|u_{|\omega'}\|_{W^{1,p}(\omega')} \leq c\|u\|_{H^1(\omega)} \qquad \forall u \in \mathcal{H}.$$

*Proof.* First, we observe that, as the kernel of a continuous linear mapping, $\mathcal{H}$ is a closed subspace of $H^1(\omega)$. Hence $\mathcal{H}$ endowed with the $H^1(\omega)$ norm is a Banach space. We consider now a sequence $(u_n)_{n\in\mathbb{N}}$ of elements of $\mathcal{H}$ such that $\lim_{n\to+\infty}\|u_n-u\|_{H^1(\omega)}=0$ and $\lim_{n\to+\infty}\|u_{n|\omega'}-v\|_{W^{1,p}(\omega')}=0$ for some $(u,v)\in\mathcal{H}\times W^{1,p}(\omega')$. Using that the $W^{1,p}(\omega')$ norm is finer than the $H^1(\omega')$ norm, we obtain that $v=u_{|\omega'}$. Hence the linear mapping $u\in\mathcal{H}\mapsto u_{|\omega'}\in W^{1,p}(\omega')$ has a closed graph. The result follows straightforwardly by application of the closed graph theorem. ∎

LEMMA 5.2. *For all $\eta\in I$, the mapping*

$$\begin{array}{rcl} W^{1,\infty}(\mathbb{R}^2,\mathbb{R}^2) & \to & \mathbb{R}, \\ V & \mapsto & G(\Omega(V))(\eta) \end{array}$$

*with $\Omega(V)=(I+V)(\Omega)$ is of class $\mathcal{C}^\infty$ in a neighborhood of* $0$.

*Proof.* We set $u_V=u_{\Omega(V)}(\eta)\circ(I+V)$. By the change of variable $x\mapsto(I+V)^{-1}(x)$ we obtain that, for $\|V\|_{W^{1,\infty}(D)}$ sufficiently small, $u_V\in\mathcal{V}$ solves

$$\int_D A(V)\nabla u_V.\nabla v dx=\langle F(\eta),v\rangle \qquad \forall v\in\mathcal{V},$$

with

$$A(V)=\alpha_\Omega \mathrm{Jac}(I+V)D(I+V)^{-1}D(I+V)^{-T}.$$

By the implicit function theorem, the mapping $A(V)\in L^\infty(D,\mathcal{M}_2(\mathbb{R}))\mapsto u_V\in\mathcal{V}$ is of class $\mathcal{C}^\infty$ in a neighborhood of $A(0)=\alpha_\Omega I$. Moreover, the mapping $V\in W^{1,\infty}(\mathbb{R}^2,\mathbb{R}^2)\mapsto A(V)\in L^\infty(\mathbb{R}^2,\mathcal{M}_2(\mathbb{R}))$ is of class $\mathcal{C}^\infty$ in a neighborhood of 0 [20]. Then the composite mapping $V\in W^{1,\infty}(\mathbb{R}^2,\mathbb{R}^2)\mapsto G(\Omega(V))(\eta)=\langle F(\eta),u_V(\eta)\rangle-M(\eta)$ is of class $\mathcal{C}^\infty$ in a neighborhood of 0. ∎

**5.2.3. Shape and topological sensitivities at a fixed load.** In this section we fix some $\eta\in I$. For notational simplicity, we drop the dependence on $\eta$ and denote the load by $\varphi:=F(\eta)$. Also, we write $\Omega_{t,\rho}:=\Omega(tV,z,\rho)$, $u_{t,\rho}:=u_{\Omega(tV,z,\rho)}$, $\alpha_{t,\rho}:=\alpha_{\Omega(tV,z,\rho)}$. We have for all $(t,\rho)\in[0,t_0]\times[0,\rho_0]^m$,

$$\int_D \alpha_{t,\rho}\nabla u_{t,\rho}.\nabla v dx=\langle\varphi,v\rangle \qquad \forall v\in\mathcal{V}, \tag{5.3}$$

$$G(\Omega_{t,\rho})=\int_D \alpha_{t,\rho}|\nabla u_{t,\rho}|^2dx-M=\langle\varphi,u_{t,\rho}\rangle-M.$$

We shall calculate the first variation of this functional with respect to the pair $(t,\rho)$. Exploiting the fact that this is a self-adjoint problem, we proceed as follows:

$$\begin{aligned} G(\Omega_{t,\rho})-G(\Omega)&=\langle\varphi,u_{t,\rho}\rangle-\langle\varphi,u_{0,0}\rangle \\ &=\int_D \alpha_{0,0}\nabla u_{0,0}.\nabla u_{t,\rho}dx-\int_D \alpha_{t,\rho}\nabla u_{t,\rho}.\nabla u_{0,0}dx \\ &=\int_D(\alpha_{0,0}-\alpha_{t,\rho})\nabla u_{0,0}.\nabla u_{t,\rho}dx. \end{aligned}$$

We define the variation

$$\tilde{\alpha}_{t,\rho}=\alpha_{t,\rho}-\alpha_{0,0}.$$

For simplicity, we denote $(0,\dots,0,\rho_i,0,\dots,0)$ by $\rho_i$. The assumptions made imply clearly that, for all $(t,\rho)\in[0,t_0]\times[0,\rho_0]^m$,

$$\tilde{\alpha}_{t,\rho}=\tilde{\alpha}_{t,0}+\tilde{\alpha}_{0,\rho}=\tilde{\alpha}_{t,0}+\sum_{i=1}^{m}\tilde{\alpha}_{0,\rho_i}. \tag{5.4}$$

This entails

$$G(\Omega_{t,\rho})-G(\Omega)=-\int_D\tilde{\alpha}_{t,0}\nabla u_{0,0}.\nabla u_{t,\rho}dx-\sum_{i=1}^{m}\int_D\tilde{\alpha}_{0,\rho_i}\nabla u_{0,0}.\nabla u_{t,\rho}dx.$$

Splitting $u_{t,\rho}$ into $u_{t,0}+(u_{t,\rho}-u_{t,0})$ in the first integral and into $u_{0,\rho_i}+(u_{t,\rho}-u_{0,\rho_i})$ in the second integral yields

$$\begin{aligned}G(\Omega_{t,\rho})-G(\Omega)=&-\int_D\tilde{\alpha}_{t,0}\nabla u_{0,0}.\nabla u_{t,0}dx+\int_D\tilde{\alpha}_{t,0}\nabla u_{0,0}.(\nabla u_{t,0}-\nabla u_{t,\rho})dx\\&-\sum_{i=1}^{m}\int_D\tilde{\alpha}_{0,\rho_i}\nabla u_{0,0}.\nabla u_{0,\rho_i}dx+\sum_{i=1}^{m}\int_D\tilde{\alpha}_{0,\rho_i}\nabla u_{0,0}.(\nabla u_{0,\rho_i}-\nabla u_{t,\rho})dx.\end{aligned}$$

Then replacing $\tilde{\alpha}_{t,0}$ and $\tilde{\alpha}_{0,\rho_i}$ by their definitions in the first and third integrals, respectively, provides

$$\begin{aligned}G(\Omega_{t,\rho})-G(\Omega)=&-\int_D\alpha_{t,0}\nabla u_{0,0}.\nabla u_{t,0}dx+\int_D\alpha_{0,0}\nabla u_{0,0}.\nabla u_{t,0}dx\\&+\int_D\tilde{\alpha}_{t,0}\nabla u_{0,0}.(\nabla u_{t,0}-\nabla u_{t,\rho})dx\\&+\sum_{i=1}^{m}\left[-\int_D\alpha_{0,\rho_i}\nabla u_{0,0}.\nabla u_{0,\rho_i}dx+\int_D\alpha_{0,0}\nabla u_{0,0}.\nabla u_{0,\rho_i}dx\right]\\&+\sum_{i=1}^{m}\int_D\tilde{\alpha}_{0,\rho_i}\nabla u_{0,0}.(\nabla u_{0,\rho_i}-\nabla u_{t,\rho})dx.\end{aligned}$$

By using (5.3) we obtain

$$\begin{aligned}G(\Omega_{t,\rho})-G(\Omega)=&-\langle\varphi,u_{0,0}\rangle+\langle\varphi,u_{t,0}\rangle+\int_D\tilde{\alpha}_{t,0}\nabla u_{0,0}.(\nabla u_{t,0}-\nabla u_{t,\rho})dx\\&+\sum_{i=1}^{m}\Big[-\langle\varphi,u_{0,0}\rangle+\langle\varphi,u_{0,\rho_i}\rangle\Big]+\sum_{i=1}^{m}\int_D\tilde{\alpha}_{0,\rho_i}\nabla u_{0,0}.(\nabla u_{0,\rho_i}-\nabla u_{t,\rho})dx.\end{aligned}$$

We arrive at

$$G(\Omega_{t,\rho})-G(\Omega)=\big[G(\Omega_{t,0})-G(\Omega)\big]+\sum_{i=1}^{m}\big[G(\Omega_{0,\rho_i})-G(\Omega)\big]+E_0(t,\rho)+\sum_{i=1}^{m}E_i(t,\rho) \tag{5.5}$$

with

$$E_0(t,\rho)=\int_D\tilde{\alpha}_{t,0}\nabla u_{0,0}.(\nabla u_{t,0}-\nabla u_{t,\rho})dx, \tag{5.6}$$

and, for all $i = 1, \ldots, m$,

$$E_i(t,\rho) = \int_D \tilde{\alpha}_{0,\rho_i} \nabla u_{0,0}.(\nabla u_{0,\rho_i} - \nabla u_{t,\rho}) dx. \tag{5.7}$$

We shall now estimate the remainders $E_i(t,\rho)$, $i = 0, \ldots, m$. We will denote by $c$ any constant independent of $t$ and $\rho$. We set $w_{t,\rho} = u_{t,0} - u_{t,\rho}$. It stems from (5.3) and (5.4) that

$$\int_D \alpha_{t,\rho} \nabla w_{t,\rho}.\nabla v dx = \int_D \tilde{\alpha}_{0,\rho} \nabla u_{t,0}.\nabla v dx \qquad \forall v \in \mathcal{V}.$$

We split $w_{t,\rho}$ into $w'_{t,\rho} + w''_{t,\rho}$, where $w'_{t,\rho}$ and $w''_{t,\rho}$ are the solutions in $\mathcal{V}$ of

$$\int_D \alpha_{0,\rho} \nabla w'_{t,\rho}.\nabla v dx = \int_D \tilde{\alpha}_{0,\rho} \nabla u_{t,0}.\nabla v dx \qquad \forall v \in \mathcal{V},$$

$$\int_D \alpha_{t,\rho} \nabla w''_{t,\rho}.\nabla v dx = -\int_D \tilde{\alpha}_{t,0} \nabla w'_{t,\rho}.\nabla v dx \qquad \forall v \in \mathcal{V}.$$

We have by elliptic regularity

$$\|w'_{t,\rho}\|_{H^1(D)} \le c\|\tilde{\alpha}_{0,\rho} \nabla u_{t,0}\|_{L^2(D)}.$$

As $u_{t,0}$ is of regularity $W^{1,\infty}$ in the vicinity of the inclusions, we arrive at

$$\|w'_{t,\rho}\|_{H^1(D)} \le c|\rho| := \max_{j=1,\ldots,m} \rho_j.$$

Next, we have $w'_{t,\rho} \in W^{1,p}(\Delta_\Omega)$ for some $p > 4$ ($w'_{t,\rho}$ is $H^2$ on each side of $\Gamma_\Omega$; see, e.g., [27]). Lemma 5.1 provides

$$\|w'_{t,\rho}\|_{W^{1,p}(\Delta_\Omega)} \le c\|w'_{t,\rho}\|_{H^1(D)} \le c|\rho|.$$

By elliptic regularity and the Hölder inequality we arrive at

$$\|w''_{t,\rho}\|_{H^1(D)} \le c\|\tilde{\alpha}_{t,0} \nabla w'_{t,\rho}\|_{L^2(D)} \le c\|\tilde{\alpha}_{t,0}\|_{L^{\frac{2p}{p-2}}(D)} \|\nabla w'_{t,\rho}\|_{L^p(\Delta_\Omega)} \le ct^{\frac{1}{2}-\frac{1}{p}}|\rho|.$$

Applying the Hölder inequality to (5.6) yields

$$\begin{aligned}
|E_0(t,\rho)| &\le \|\tilde{\alpha}_{t,0}\|_{L^{\frac{p}{p-2}}(\Delta_\Omega)} \|\nabla u_{0,0}\|_{L^p(\Delta_\Omega)} \|\nabla w'_{t,\rho}\|_{L^p(\Delta_\Omega)} \\
&\quad + \|\tilde{\alpha}_{t,0}\|_{L^{\frac{2p}{p-2}}(\Delta_\Omega)} \|\nabla u_{0,0}\|_{L^p(\Delta_\Omega)} \|\nabla w''_{t,\rho}\|_{L^2(\Delta_\Omega)} \\
&\le ct^{1-\frac{2}{p}}|\rho| + ct^{\frac{1}{2}-\frac{1}{p}} t^{\frac{1}{2}-\frac{1}{p}}|\rho| \\
&\le ct^{\frac{p-4}{2p}}(t + |\rho|^2).
\end{aligned}$$

Now we set $w_{t,\rho,i} = u_{0,\rho_i} - u_{t,\rho}$. We derive from (5.3) and (5.4) that

$$\int_D \alpha_{t,\rho} \nabla w_{t,\rho,i}.\nabla v dx = \int_D \left( \tilde{\alpha}_{t,0} + \sum_{j \ne i} \tilde{\alpha}_{0,\rho_j} \right) \nabla u_{0,\rho_i}.\nabla v dx \qquad \forall v \in \mathcal{V}.$$

We denote by $\bar{\rho}_i$ the vector $(\rho_1, \dots, \rho_{i-1}, 0, \rho_{i+1}, \dots, \rho_m)$. We split $w_{t,\rho,i}$ into $w'_{t,\rho,i} + w''_{t,\rho,i}$, where $w'_{t,\rho,i}$ and $w''_{t,\rho,i}$ are the solutions in $\mathcal{V}$ of

$$\int_D \alpha_{t,\bar{\rho}_i} \nabla w'_{t,\rho,i}.\nabla v dx = \int_D \left( \tilde{\alpha}_{t,0} + \sum_{j\neq i} \tilde{\alpha}_{0,\rho_j} \right) \nabla u_{0,\rho_i}.\nabla v dx \qquad \forall v \in \mathcal{V},$$

$$\int_D \alpha_{t,\rho} \nabla w''_{t,\rho,i}.\nabla v dx = -\int_D \tilde{\alpha}_{0,\rho_i} \nabla w'_{t,\rho,i}.\nabla v dx \qquad \forall v \in \mathcal{V}.$$

We have by elliptic regularity

$$\|w'_{t,\rho,i}\|_{H^1(D)} \leq c\|\tilde{\alpha}_{t,0}\nabla u_{0,\rho_i}\|_{L^2(D)} + \sum_{j\neq i} \|\tilde{\alpha}_{0,\rho_j}\nabla u_{0,\rho_i}\|_{L^2(D)}.$$

The function $u_{0,\rho_i}$ is of regularity $W^{1,p}$, for any $p > 2$, in $\Delta_\Omega$ and in the vicinity of the inclusions with indices $j \neq i$. Therefore

$$\|w'_{t,\rho,i}\|_{H^1(D)} \leq ct^{\frac{1}{2}-\frac{1}{p}} + c|\rho|^{1-\frac{2}{p}}.$$

Furthermore, denoting by $\Delta_i$ a neighborhood of $z_i$ containing the inclusion, we have $w'_{t,\rho,i} \in W^{1,\infty}(\Delta_i)$. Lemma 5.1 results in

$$\|w'_{t,\rho,i}\|_{W^{1,\infty}(\Delta_i)} \leq ct^{\frac{1}{2}-\frac{1}{p}} + c|\rho|^{1-\frac{2}{p}}.$$

Additionally, we have

$$\begin{aligned} \|w''_{t,\rho,i}\|_{H^1(D)} &\leq c\|\tilde{\alpha}_{0,\rho_i}\nabla w'_{t,\rho,i}\|_{L^2(D)} \\ &\leq c\|\tilde{\alpha}_{0,\rho_i}\|_{L^2(D)}\|\nabla w'_{t,\rho,i}\|_{L^\infty(\Delta_i)} \leq c|\rho|(t^{\frac{1}{2}-\frac{1}{p}} + |\rho|^{1-\frac{2}{p}}). \end{aligned}$$

We derive from (5.7) that

$$\begin{aligned} |E_i(t,\rho)| \leq\ & \|\tilde{\alpha}_{0,\rho_i}\|_{L^1(\Delta_i)}\|\nabla u_{0,0}\|_{L^\infty(\Delta_i)}\|\nabla w'_{t,\rho,i}\|_{L^\infty(\Delta_i)} \\ &+\|\tilde{\alpha}_{0,\rho_i}\|_{L^2(\Delta_i)}\|\nabla u_{0,0}\|_{L^\infty(\Delta_i)}\|\nabla w''_{t,\rho,i}\|_{L^2(\Delta_i)} \\ \leq\ & c|\rho|^2(t^{\frac{1}{2}-\frac{1}{p}} + |\rho|^{1-\frac{2}{p}}) + c|\rho||\rho|(t^{\frac{1}{2}-\frac{1}{p}} + |\rho|^{1-\frac{2}{p}}) \\ \leq\ & c|\rho|^2(t^{\frac{1}{2}-\frac{1}{p}} + |\rho|^{1-\frac{2}{p}}). \end{aligned}$$

We arrive at

$$G(\Omega_{t,\rho}) - G(\Omega) = \left[G(\Omega_{t,0}) - G(\Omega)\right] + \sum_{i=1}^m \left[G(\Omega_{0,\rho_i}) - G(\Omega)\right] + O(\|(t,\rho^2)\|^{1+\delta})$$

for some $\delta > 0$, with $\rho^2 := (\rho_1^2, \dots, \rho_m^2)$ and an arbitrary norm on $\mathbb{R}^{m+1}$.

Now we can use a known result on topological sensitivity (see [4]),

$$G(\Omega_{0,\rho_i}) - G(\Omega) = 2\pi s(z_i)\rho_i^2 \alpha_\Omega(z_i)\frac{\alpha^+ - \alpha^-}{\alpha^+ + \alpha^-}|\nabla u_\Omega(z_i)|^2 + O\left(\rho_i^{2+\gamma}\right),$$

with $s(z_i) = 1$ if $z_i \in \Omega$, $s(z_i) = -1$, and otherwise $\gamma > 0$. As to the shape sensitivity, it is established in [23] that

$$G(\Omega_{t,0}) - G(\Omega) = -(\alpha^+ - \alpha^-)\int_{\Gamma_\Omega}\left(\frac{1}{\alpha^+\alpha^-}|\alpha_\Omega(x)\nabla u_\Omega(x).n|^2 + |\nabla_\tau u_\Omega(x)|^2\right) tV.nds + o(t).$$

The notation $\nabla_\tau$ stands for the tangential component of the gradient. This result is also obtained in [1], provided, however, that $V$ enjoys a higher regularity. Using Lemma 5.2 and the Taylor–Lagrange inequality, it turns out that the remainder behaves actually like an $O(t^2)$. We obtain

(5.8)
$$G(\Omega_{t,\rho})-G(\Omega) = -t(\alpha^+-\alpha^-)\int_{\Gamma_\Omega}\left(\frac{1}{\alpha^+\alpha^-}|\alpha_\Omega(x)\nabla u_\Omega(x).n|^2 + |\nabla_\tau u_\Omega(x)|^2\right)V.nds \\ + 2\pi\frac{\alpha^+-\alpha^-}{\alpha^++\alpha^-}\sum_{i=1}^m s(z_i)\rho_i^2\alpha_\Omega(z_i)|\nabla u_\Omega(z_i)|^2 + O(\|(t,\rho^2)\|^{1+\delta}),$$

where $\delta$ is a positive number.

**5.2.4. Collective shape and topological sensitivities.** We shall now check that the above expansion is uniform with respect to $\varphi$ on $\mathcal{F} := F(I)$. Then it will be uniform with respect to $\eta$ on $I$. We write $\theta = (t,\rho^2) \in [0,t_0]\times[0,\rho_0]^m$ and the remainder $O(\|\theta\|^{1+\delta})$ in the form $\|\theta\|^{1+\delta}R(\theta,\varphi)$ with

$$|R(\theta,\varphi)| \le R_0(\varphi) \in \mathbb{R} \qquad \forall\varphi\in\mathcal{U},\ \forall\theta\in[0,t_0]\times[0,\rho_0]^m.$$

We observe that, as a difference of quadratic forms, $R(\theta,.)$ is a continuous quadratic form on $\mathcal{U}$ for all $\theta\in[0,t_0]\times[0,\rho_0]^m$. We denote by $B(\theta,.,.)$ the polarization of $R(\theta,.)$ defined by

$$B(\theta,\varphi,\psi) = \frac{1}{4}\left[R(\theta,\varphi+\psi) - R(\theta,\varphi-\psi)\right] \qquad \forall\varphi,\psi\in\mathcal{U}.$$

Therefore,

$$|B(\theta,\varphi,\psi)| \le \frac{1}{4}\left[R_0(\varphi+\psi)+R_0(\varphi-\psi)\right] \qquad \forall\varphi,\psi\in\mathcal{U},\ \forall\theta\in[0,t_0]\times[0,\rho_0]^m.$$

Hence, for all $\varphi,\psi\in\mathcal{U}$, the family $(B(\theta,\varphi,\psi))_{\theta\in[0,t_0]\times[0,\rho_0]^m}$ is bounded. By the Banach–Steinhaus theorem, there exists a constant $c(\varphi)$ such that

$$\|B(\theta,\varphi,.)\|_{\mathcal{U}'} \le c(\varphi) \qquad \forall\theta\in[0,t_0]\times[0,\rho_0].$$

Applying once again the Banach–Steinhaus theorem results in

$$\sup_{\theta\in[0,t_0]\times[0,\rho_0]^m}\|\varphi\mapsto B(\theta,\varphi,.)\|_{\mathcal{L}(\mathcal{U},\mathcal{U}')} < +\infty.$$

In other words, there exists $c>0$ such that

$$|B(\theta,\varphi,\psi)| \le c\|\varphi\|_{\mathcal{U}}\|\psi\|_{\mathcal{U}} \qquad \forall\theta\in[0,t_0]\times[0,\rho_0]^m,\ \forall\varphi,\psi\in\mathcal{U}.$$

Taking $\psi=\varphi$ entails

$$|R(\theta,\varphi)| \le c\|\varphi\|_{\mathcal{U}}^2 \qquad \forall\theta\in[0,t_0]\times[0,\rho_0]^m,\ \forall\varphi\in\mathcal{U}.$$

Using the boundedness of $\mathcal{F}$, we arrive at

$$\sup_{\theta\in[0,t_0]\times[0,\rho_0]^m} \sup_{\varphi\in\mathcal{F}} |R(\theta,\varphi)| < +\infty.$$

Hence we can write the asymptotic expansion in the sense of the norm of $Y$:

$$\begin{aligned}(5.9)\quad & G(\Omega_{t,\rho}) - G(\Omega)\\ &= -t(\alpha^+ - \alpha^-)\int_{\Gamma_\Omega}\left(\frac{1}{\alpha^+\alpha^-}|\alpha_\Omega(x)\nabla u_\Omega(.)(x).n|^2 + |\nabla_\tau u_\Omega(.)(x)|^2\right)V.nds\\ &+ 2\pi\frac{\alpha^+ - \alpha^-}{\alpha^+ + \alpha^-}\sum_{i=1}^m s(z_i)\rho_i^2\alpha_\Omega(z_i)|\nabla u_\Omega(.)(z_i)|^2 + O_Y(\|\theta\|^{1+\delta}).\end{aligned}$$

Therefore we obtain (3.4) with

$$f(\rho) = \pi\rho^2,$$

$$G'_S(\Omega)(\eta)(V) = -\int_{\Gamma_\Omega}(\alpha^+ - \alpha^-)\left(\frac{1}{\alpha^+\alpha^-}|\alpha_\Omega\nabla u_\Omega(\eta).n|^2 + |\nabla_\tau u_\Omega(\eta)|^2\right)(V.n)ds,$$

$$G'_T(\Omega)(\eta) = 2\frac{\alpha^+ - \alpha^-}{\alpha^+ + \alpha^-}s\alpha_\Omega|\nabla u_\Omega(\eta)|^2.$$

We recall that the sign function $s$ takes the value $+1$ in $\Omega$ and $-1$ outside. For the objective functional we have obviously

$$\langle J'_S(\Omega), V\rangle = \int_{\Gamma_\Omega}(V.n)ds, \qquad J'_T(\Omega) = -s.$$

**5.3. Optimality conditions.** From inspection of the expressions of the shape and topological derivatives, it appears that the constraint qualification condition is fulfilled in each of the following cases. Another case where this condition is assured is discussed in Remark 3.

- The interface $\Gamma_\Omega$ is smooth and there exists $x_0 \in \Gamma_\Omega \cap \tilde{D}$, $\varepsilon > 0$, and $r > 0$ such that

$$\frac{1}{\alpha^+\alpha^-}|\alpha_\Omega\nabla u_\Omega(\eta)(x).n|^2 + |\nabla_\tau u_\Omega(\eta)(x)|^2 \geq \varepsilon \quad \forall x \in \Gamma_\Omega \cap \tilde{D} \cap \mathbb{B}(x_0, r),\ \forall\eta \in I.$$

- There exists $x_0 \in D\backslash\overline{\Omega}$ (if $\alpha_+ > \alpha_-$; otherwise $x_0 \in \Omega$) such that $\nabla u_\Omega(\eta)(x_0) \neq 0$ for all $\eta \in I$.

Then the necessary optimality conditions read as follows: there exists $\mu \in K^+$, the set of positive Radon measures on $I$, such that

$$\int_I G(\Omega)(\eta)d\mu(\eta) = 0,$$

$$1 - g_S(\Omega,\mu) = 0 \ \text{ on } \Gamma_\Omega \cap \tilde{D}, \tag{5.10}$$

$$1 - g_T(\Omega,\mu) \leq 0 \ \text{ in } \Omega, \tag{5.11}$$

$$1 - g_T(\Omega,\mu) \geq 0 \ \text{ in } D \setminus \overline{\Omega}, \tag{5.12}$$

with

(5.13)
$$g_S(\Omega,\mu)(x) = \int_I (\alpha^+ - \alpha^-)\left(\frac{1}{\alpha^+\alpha^-}|\alpha_\Omega(x)\nabla u_\Omega(\eta)(x).n|^2 + |\nabla_\tau u_\Omega(\eta)(x)|^2\right) d\mu(\eta),$$

$$g_T(\Omega,\mu)(x) = \int_I 2\frac{\alpha^+ - \alpha^-}{\alpha^+ + \alpha^-}\alpha_\Omega(x)|\nabla u_\Omega(\eta)(x)|^2 d\mu(\eta). \tag{5.14}$$

We denote by $g_T^+(\Omega,\mu)$ (resp., $g_T^-(\Omega,\mu)$) the restriction of $g_T(\Omega,\mu)$ to $\Omega$ (resp., $D\setminus\overline{\Omega}$). By continuity of $\alpha_\Omega\nabla u_\Omega(\eta).n$ and $\nabla_\tau u_\Omega(\eta)$ across $\Gamma_\Omega\cap\tilde{D}$, we obtain the equality in the sense of traces

$$g_S(\Omega,\mu) = \frac{g_T^+(\Omega,\mu) + g_T^-(\Omega,\mu)}{2} \quad \text{on } \Gamma_\Omega\cap\tilde{D}.$$

Conditions (5.10)–(5.12) can be rewritten

$$g_T^+(\Omega,\mu) \geq 1 \quad \text{in } \Omega, \tag{5.15}$$

$$g_T^-(\Omega,\mu) \leq 1 \quad \text{in } D\setminus\overline{\Omega}, \tag{5.16}$$

$$\frac{g_T^+(\Omega,\mu) + g_T^-(\Omega,\mu)}{2} = 1 \quad \text{on } \Gamma_\Omega\cap\tilde{D}. \tag{5.17}$$

*Remark* 3. The constraint qualification can be ascertained, for instance, in the case where $\Omega \subset\subset \tilde{D}$ (the same argument applies if $D\setminus\overline{\Omega}\subset\subset\tilde{D}$), with $\Omega\neq\emptyset$, $\partial\Omega$ smooth, and $0\notin F(I)$. Indeed, assume that, for some $\eta\in I$, $\frac{1}{\alpha^+\alpha^-}|\alpha_\Omega\nabla u_\Omega(\eta).n|^2 + |\nabla_\tau u_\Omega(\eta)|^2 = 0$ on $\partial\Omega$. As $\nabla u_\Omega(\eta).n$ vanishes on each side of $\partial\Omega$, $u_\Omega(\eta)$ is harmonic in $D$. In addition, $u_\Omega(\eta)$ is constant on each connected component of $\partial\Omega$. Denoting by $\partial\omega$ one of these connected components, where $\omega$ is some subdomain of $D$, we have by uniqueness that $u_\Omega(\eta)$ is constant in $\omega$. By analyticity, $u_\Omega(\eta)$ is constant in $D$; hence $\nabla u_\Omega(\eta).n = 0$ on $\Gamma_N$, which is excluded. Thus, choosing $V$ as a lifting of the normal to $\partial\Omega$ corrected with a minus sign if $\alpha^+ < \alpha^-$, we have $G'_S(\Omega)(\eta)(V) < 0$ for all $\eta\in I$. By continuity with respect to $\eta$ and compactness of $I$, this implies that $G(\Omega) + G'_S(\Omega)(V) \in \text{Int}(-K)$.

*Remark* 4. The conditions (5.15)–(5.17) pass to the limit when the background density $\alpha^-$ goes to zero. This amounts to considering a Neumann boundary condition on $\Gamma_\Omega$ and around the inclusions, which in this case are only holes ($\mathcal{T}(\Omega)=\Omega$). The additivity of the shape and topological sensitivities at a fixed load is proved in [29] (under slightly different regularity assumptions), and the expressions of the shape and topological derivatives are well known [4, 23, 28, 30]. Then the optimality conditions are the following: there exists $\mu\in K^+$ such that $\int_I G(\Omega)(\eta)d\mu(\eta) = 0$ and

$$g_T(\Omega,\mu) \geq 1 \quad \text{in } \Omega, \tag{5.18}$$

$$g_T(\Omega,\mu) = 2 \quad \text{on } \Gamma_\Omega\cap\tilde{D}, \tag{5.19}$$

with

$$g_T(\Omega,\mu)(x) = \int_I 2\alpha^+|\nabla u_\Omega(\eta)(x)|^2 d\mu(\eta).$$

These conditions generalize those stated in [13] for a single load (thus a single constraint).

**6. Third example.** Let $D$ be a bounded domain of $\mathbb{R}^2$ with a Lipschitz boundary $\partial D$, and let $\mathcal{Q}$ be a closed subset of $D$. We denote by $\Gamma$ a part of $\partial D$ with nonzero measure. We choose $\mathscr{E}$ as the set of all Lipschitz subdomains $\Omega$ of $D$ containing $\mathcal{Q}$ and such that $\Gamma \subset \partial\Omega$ (see Figure 3). We are given a distribution $\psi \in H^{-1/2}(\Gamma)$ with $\langle \psi, 1\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)} = 0$. For all $\Omega \in \mathscr{E}$, we denote by $u_\Omega$ the solution of the boundary value problem

$$\left\{\begin{array}{rcll} -\Delta u_\Omega &=& 0 & \text{in} \quad \Omega, \\ \partial_n u_\Omega &=& \psi & \text{on} \quad \Gamma, \\ \partial_n u_\Omega &=& 0 & \text{on} \quad \partial\Omega \setminus \Gamma, \end{array}\right. \tag{6.1}$$

satisfying the normalization condition $\int_{\partial\Omega} u_\Omega ds = 0$. We deal only with topology perturbations ($\mathcal{S}(\Omega) = \emptyset$) located in $\mathcal{T}(\Omega) = \mathcal{H} \cap \Omega$, where $\mathcal{H}$ is an open subset of $D$ such that $\mathcal{Q} \cap \overline{\mathcal{H}} = \emptyset$ and $\Gamma \cap \overline{\mathcal{H}} = \emptyset$. We consider an objective functional $J(\Omega)$ to be minimized which admits a topological asymptotic expansion of the following form: for all $m \in \mathbb{N}^*$, $z \in \mathcal{T}(\Omega)^{[m]}$, $\rho \in \mathbb{R}_+^m$, $\varepsilon \to 0^+$,

$$J(\Omega \setminus \cup_{i=1}^m \overline{\mathbb{B}(z_i, \varepsilon\rho_i)}) = J(\Omega) + \pi\varepsilon^2 \sum_{i=1}^m \rho_i^2 J'_T(\Omega)(z_i) + o(\varepsilon^2).$$

Given a constant $M \in \mathbb{R}$, we impose the constraint

$$u_\Omega(x) \le M \quad \forall x \in \mathcal{Q}.$$

Therefore we set $Y = \mathcal{C}(\mathcal{Q})$, $K = \{\varphi \in Y, \varphi(x) \ge 0 \ \forall x \in \mathcal{Q}\}$, $G(\Omega) = (u_\Omega - M)_{|\mathcal{Q}}$. For all $y \in \mathcal{Q}$, we have the asymptotic expansion (see [18, 3])

$$u_{\Omega(z,\varepsilon\rho)}(y) = u_\Omega(y) - \pi\varepsilon^2 \sum_{i=1}^m \rho_i^2 \nabla_z N(z_i, y).\nabla u_\Omega(z_i) + o(\varepsilon^2),$$

where $N(., y)$ is the Neumann function, i.e., the solution of

$$\left\{\begin{array}{rcll} -\Delta_x N(x,y) &=& \delta_y & \text{in} \quad \Omega, \\ \partial_{n_x} N(x,y) &=& -\dfrac{1}{|\partial\Omega|} & \text{on} \quad \partial\Omega, \end{array}\right. \tag{6.2}$$

verifying $\int_{\partial\Omega} N(x,y)ds(x) = 0$. In addition, the remainder $o(\varepsilon^2)$ is uniform with respect to $y \in \mathcal{Q}$. Thus we can write

$$G(\Omega \setminus \cup_{i=1}^m \overline{\mathbb{B}(z_i, \varepsilon\rho_i)}) = G(\Omega) + \pi\varepsilon^2 \sum_{i=1}^m \rho_i^2 G'_T(\Omega)(z_i) + o(\varepsilon^2),$$

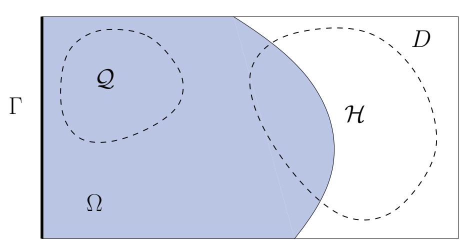


FIG. 3. *Geometrical configuration for the third example.*

with

$$G'_T(\Omega)(z) = \left\{ \begin{array}{lll} \mathcal{Q} & \to & \mathbb{R}, \\ y & \mapsto & -\nabla_z N(z,y).\nabla u_\Omega(z). \end{array} \right.$$

The Neumann function can be split into (see [3])

$$N(x,y) = -\frac{1}{2\pi}\ln|x-y| + R(x,y), \tag{6.3}$$

where, for all $y \in \Omega$, the function $R(.,y)$ solves

$$\left\{ \begin{array}{lllll} -\Delta_x R(x,y) & = & 0 & \text{in} & \Omega, \\ \partial_{n_x} R(x,y) & = & -\dfrac{1}{|\partial\Omega|} + \dfrac{1}{2\pi}\dfrac{(x-y).n_x}{|x-y|^2} & \text{on} & \partial\Omega, \end{array} \right.$$

with $\int_{\partial\Omega} R(x,y)ds(x) = \frac{1}{2\pi}\int_{\partial\Omega} \ln|x-y|ds(x)$. It follows by elliptic regularity that the map $y \in \mathcal{Q} \mapsto R(.,y) \in H^1(\Omega)$ is continuous. Arguing as in the proof of Lemma 5.1, we have that, for any open set $\mathcal{A}_\Omega \subset\subset \Omega$, the map $y \in \mathcal{Q} \mapsto R(.,y) \in \mathcal{C}^1(\mathcal{A}_\Omega)$ is continuous too. Therefore, for all $z \in \mathcal{H} \cap \Omega$, the map $y \in \mathcal{Q} \mapsto \nabla_z R(z,y) \in \mathbb{R}$ is continuous, and hence $G'_T(\Omega)(z) \in Y$. The constraint qualification condition requires the existence of $m \in \mathbb{N}^*$, $(z_1,\ldots,z_m) \in (\mathcal{H} \cap \Omega)^{[m]}$, $(\beta_1,\ldots,\beta_m) \in \mathbb{R}_+^m$ such that

$$u_\Omega(y) - M - \sum_{i=1}^m \beta_i \nabla_z N(z_i,y).\nabla u_\Omega(z_i) < 0 \qquad \forall y \in \mathcal{Q}.$$

This condition may be restrictive in some circumstances. In such situations it can be useful to allow boundary perturbations. However, we focus here on topology perturbations for simplicity. Then necessary optimality conditions are as follows: there exists $\mu \in K^+$ (the set of positive Radon measures on $\mathcal{Q}$) such that

$$J'_T(\Omega)(z) - \int_{\mathcal{Q}} \nabla_z N(z,y).\nabla u_\Omega(z) d\mu(y) \geq 0 \qquad \forall z \in \mathcal{H} \cap \Omega, \tag{6.4}$$

$$\int_{\mathcal{Q}} (u_\Omega - M) d\mu = 0. \tag{6.5}$$

Equivalently, (6.5) can be rephrased as

$$u_\Omega = M \text{ on } \operatorname{supp}(\mu).$$

Next, we set

$$v_\mu(z) = \int_{\mathcal{Q}} N(z,y) d\mu(y),$$

so that (6.4) can be rewritten as

$$J'_T(\Omega)(z) - \nabla u_\Omega(z).\nabla v_\mu(z) \geq 0 \qquad \forall z \in \mathcal{H} \cap \Omega.$$

We now give a practical way of computing $v_\mu$. First, in view of the splitting (6.3) along with the bound $\sup_{y\in\mathcal{Q}} \|R(.,y)\|_{H^1(\Omega)} < \infty$ resulting from the compactness of $\mathcal{Q}$, we have

$$\sup_{y\in\mathcal{Q}} \int_\Omega |N(z,y)|^2 dz < \infty; \quad \text{hence} \quad \int_{\mathcal{Q}} \left( \int_\Omega |N(z,y)|^2 dz \right) d\mu(y) < \infty.$$

Therefore, $v_\mu \in L^2(\Omega)$ by virtue of the Fubini theorem. Similarly, we get that $v_\mu \in H^1(\mathcal{B}_\Omega \cap \Omega)$, where $\mathcal{B}_\Omega$ is any open neighborhood of $\partial\Omega$ such that $\overline{\mathcal{B}_\Omega} \cap \mathcal{Q} = \emptyset$. Then, we take an arbitrary test function $\varphi \in H^2(\Omega)$. By the Fubini theorem, we have

$$\int_\Omega v_\mu(z)\Delta\varphi(z)dz = \int_{\mathcal{Q}} \left[ \int_\Omega N(z,y)\Delta\varphi(z)dz \right] d\mu(y).$$

Yet, we derive from (6.2) that

$$\int_\Omega N(z,y)\Delta\varphi(z)dz = -\varphi(y) + \int_{\partial\Omega} N(z,y)\partial_n\varphi(z)ds(z) + \frac{1}{|\partial\Omega|}\int_{\partial\Omega}\varphi(z)ds(z),$$

which entails

$$\int_\Omega v_\mu(z)\Delta\varphi(z)dz = -\int_{\mathcal{Q}}\varphi(y)d\mu(y) + \int_{\partial\Omega} v_\mu\partial_n\varphi ds + \frac{\mu(\mathcal{Q})}{|\partial\Omega|}\int_{\partial\Omega}\varphi(z)ds(z).$$

In other words, $v_\mu$ is a very weak solution of the boundary value problem

$$\left\{ \begin{array}{rcll} -\Delta v_\mu &=& \mu\chi_{\mathcal{Q}} & \text{in} \quad \Omega, \\ \partial_n v_\mu &=& -\frac{\mu(\mathcal{Q})}{|\partial\Omega|} & \text{on} \quad \partial\Omega, \\ \int_{\partial\Omega} v_\mu ds &=& 0, & \end{array} \right.$$

where $\chi_{\mathcal{Q}}$ is the characteristic function of $\mathcal{Q}$. The function $-v_\mu$ can be interpreted as the adjoint state for the scalarized constraint functional. Its very weak regularity ($L^2$) is a well-known feature in the theory of control with pointwise state constraints (see, e.g., [22]).

## REFERENCES


[1] L. Afraites, M. Dambrine, and D. Kateb, *Shape methods for the transmission problem with a single measurement*, Numer. Funct. Anal. Optim., 28 (2007), pp. 519–551.

[2] G. Allaire, *Shape Optimization by the Homogenization Method*, Appl. Math. Sci. 146, Springer-Verlag, New York, 2002.

[3] H. Ammari and H. Kang, *Reconstruction of Small Inhomogeneities from Boundary Measurements*, Lecture Notes in Math. 1846, Springer-Verlag, Berlin, 2004.

[4] S. Amstutz, *Sensitivity analysis with respect to a local perturbation of the material property*, Asymptot. Anal., 49 (2006), pp. 87–108.

[5] S. Amstutz, *Topological sensitivity analysis for some nonlinear PDE system*, J. Math. Pures Appl., 85 (2006), pp. 540–557.

[6] S. Amstutz, *Augmented Lagrangian for cone constrained topology optimization*, Comput. Optim. Appl., to appear.

[7] S. Amstutz and H. Andrä, *A new algorithm for topology optimization using a level-set method*, J. Comput. Phys., 216 (2006), pp. 573–588.

[8] V. Barbu and Th. Precupanu, *Convexity and Optimization in Banach Spaces*, 2nd ed., Math. Appl. (East European Ser.) 10, D. Reidel Publishing Co., Dordrecht, 1986.

[9] M. P. Bendsøe and N. Kikuchi, *Generating optimal topologies in structural design using a homogenization method*, Comput. Methods Appl. Mech. Engrg., 71 (1988), pp. 197–224.

[10] M. P. Bendsøe and O. Sigmund, *Topology Optimization. Theory, Methods, Applications*, Springer-Verlag, Berlin, 2003.

[11] J. F. Bonnans and A. Shapiro, *Perturbation Analysis of Optimization Problems*, Springer Ser. Oper. Res., Springer-Verlag, New York, 2000.

[12] M. Burger and R. Stainko, *Phase-field relaxation of topology optimization with local stress constraints*, SIAM J. Control Optim., 45 (2006), pp. 1447–1466.

[13] J. Céa, S. Garreau, P. Guillaume, and M. Masmoudi, *The shape and topological optimizations connection*, Comput. Methods Appl. Mech. Engrg., 188 (2000), pp. 713–726.

[14] F. de Gournay, G. Allaire, and F. Jouve, *Shape and topology optimization of the robust compliance via the level set method*, ESAIM Control Optim. Calc. Var., 14 (2008), pp. 43–70.
[15] P. Duysinx and M. P. Bendsøe, *Topology optimization of continuum structures with local stress constraints*, Internat. J. Numer. Methods Engrg., 43 (1998), pp. 1453–1478.
[16] H. Eschenauer, V. V. Kobelev, and A. Schumacher, *Bubble method for topology and shape optimization of structures*, Struct. Multidiscip. Optim., 8 (1994), pp. 42–51.
[17] H. A. Eschenauer and A. Schumacher, *Topology and shape optimization procedures using hole positioning criteria—theory and applications*, in Topology Optimization in Structural Mechanics, CISM Courses and Lectures 374, Springer, Vienna, 1997, pp. 135–196.
[18] A. Friedman and M. Vogelius, *Identification of small inhomogeneities of extreme conductivity by boundary measurements: A theorem on continuous dependence*, Arch. Rational Mech. Anal., 105 (1989), pp. 299–326.
[19] S. Garreau, P. Guillaume, and M. Masmoudi, *The topological asymptotic for PDE systems: The elasticity case*, SIAM J. Control Optim., 39 (2001), pp. 1756–1778.
[20] A. Henrot and M. Pierre, *Variation et optimisation de formes*, Math. Appl. (Berlin) 48, Springer, Berlin, 2005.
[21] M. Hinze, R. Pinnau, M. Ulbrich, and S. Ulbrich, *Optimization with PDE Constraints*, Math. Model. Theory Appl. 23, Springer, New York, 2009.
[22] K. Ito and K. Kunisch, *Semi-smooth Newton methods for state-constrained optimal control problems*, Systems Control Lett., 50 (2003), pp. 221–228.
[23] K. Ito, K. Kunisch, and G. H. Peichl, *Variational approach to shape derivatives*, ESAIM Control Optim. Calc. Var., 14 (2008), pp. 517–539.
[24] F. Murat and J. Simon, *Quelques résultats sur le contrôle par un domaine géométrique*, Tech. report, Publ. Laboratoire d'Analyse Numérique, Université Paris 6, Paris, France, 1974.
[25] S. A. Nazarov and J. Sokołowski, *Asymptotic analysis of shape functionals*, J. Math. Pures Appl. (9), 82 (2003), pp. 125–196.
[26] B. Samet, S. Amstutz, and M. Masmoudi, *The topological asymptotic for the Helmholtz equation*, SIAM J. Control Optim., 42 (2003), pp. 1523–1544.
[27] G. Savaré, *Regularity results for elliptic equations in Lipschitz domains*, J. Funct. Anal., 152 (1998), pp. 176–201.
[28] J. Sokołowski and A. Żochowski, *On the topological derivative in shape optimization*, SIAM J. Control Optim., 37 (1999), pp. 1251–1272.
[29] J. Sokołowski and A. Żochowski, *Optimality conditions for simultaneous topology and shape optimization*, SIAM J. Control Optim., 42 (2003), pp. 1198–1221.
[30] J. Sokołowski and J.-P. Zolésio, *Introduction to Shape Optimization, Shape Sensitivity Analysis,* Springer Ser. Comput. Math., 16, Springer-Verlag, Berlin, 1992.