Published in Optim. Methods Softw. 26 (2011), no. 4-5, 555–573

ANALYSIS OF A LEVEL SET METHOD FOR TOPOLOGY OPTIMIZATION

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ABSTRACT. A level set method for the topological optimization of domains is proposed and analyzed in this paper. The framework of the analysis is that of elliptic optimal control problems with a binary control acting as a source term of the state equation. The main ingredient of the algorithm is the concept of topological derivative. Convergence results are proven and illustrated by some numerical experiments.

1. Introduction

The representation of free boundaries as the zero level set of a function defined over a larger fixed domain has become very popular in shape optimization. From the numerical point of view, a major advantage of this strategy is to allow an accurate description of the boundaries on a fixed mesh, thus avoiding costly mesh adaptation or remeshing procedures. Also, as the sign of the so-called level set function permits to clearly segment the hold-all into two regions, no intermediate densities need to be introduced. This approach is therefore in this respect more direct than relaxation methods, where penalization techniques must be applied in order to enforce extremal densities (see, *e.g.*, [1, 6, 7]).

Since the pioneering work [19] dedicated to the numerical tracking of fronts, most authors employ a Hamilton-Jacobi type equation to govern the evolution of the level set function. This formulation arises naturally when one wants to move the level sets according to a prescribed velocity field (see, *e.g.*, [18] for a review on level set methods). The same idea has been applied in shape optimization [3, 22], with the help of the following ideas: a fictitious time variable is involved, the opposite of the shape derivative is used as velocity field, and the empty region is replaced by a weak phase in order to avoid the singularity of the stiffness matrix and to extend the shape derivative to the whole computational domain. Nevertheless this approach has some drawbacks. Firstly, due to the discontinuity of the velocity field at the interface, the level set function tends to become very steep in this place, which necessitates periodic reinitializations. Secondly, the method is not designed to produce topology variations. In fact, holes can merge or cancel, but cannot nucleate. Hence the obtained domains can be very dependent on the initial guess. Finally, the numerical solution of the Hamilton-Jacobi equation involves a CFL condition which makes the propagation of the interface rather slow. Several cures to these inconveniences have been devised, leading to a number of variants of the original method [2, 8, 9, 11, 14, 16, 17].

A level-set-based method especially designed to allow topology variations was proposed in [5]. The main ingredient is the notion of topological derivative [4, 12, 15, 21], which measures the sensitivity of the objective functional with respect to the nucleation of small holes within the domain. Unlike the original method and its variants, this one does not rely on a Hamilton-Jacobi formulation. Instead, topological optimality conditions are written in the form of a fixed point relation, which is then solved by successive approximations. All kinds of shape and topology variations can therefore occur without introducing any parameter to decide how to combine descent directions of different natures. The present paper is dedicated to the convergence analysis of this method. For simplicity, we consider the situation where the shape intervenes at the right hand side of the state equation. In this case, the topological derivative does not depend on the shape of the holes, hence it can be more easily exploited to account for general domain variations.

The paper is organized as follows. The problem under consideration is stated in Section 2. Optimality conditions are derived in Section 3. The algorithm is described in section 4, and its convergence analysis is carried out in Sections 5 through 7. The discrete version of the algorithm is specified in

2000 *Mathematics Subject Classification.* 49K20,49M05,49M25,49Q10,49Q12.

Key words and phrases. topology optimization, level set method, topological sensitivity, topological derivative.

Section 8, and some numerical examples are shown in Section 9. An appendix collects some auxiliary results.

2. Problem statement

Let D be an open and bounded subset of $\mathbb{R}^2$ with Lipschitz boundary ∂D. Given a pair $(u^-, u^+) \in \mathbb{R}^2$ with $u^+ < u^-$ (this choice of notation will be justified later), we define $\mathcal{E}$ as the set of all measurable functions

$$u : D \to \{u^-, u^+\}.$$

Let now Γ_D be a part of nonzero measure of ∂D and $\mathcal{V}$ be the set of all functions of $H^1(D)$ with zero trace on Γ_D. Given a function $y^\dagger \in L^2(D)$ and a coefficient $\nu \in \mathbb{R}$, we investigate the following model problem:

$$\underset{(u,y)\in\mathcal{E}\times\mathcal{V}}{\text{Minimize}}\, J(u,y) = \frac{1}{2}\int_D (y - y^\dagger)^2 dx + \nu \int_D u dx \tag{2.1}$$

subject to the Poisson equation

$$\int_D \nabla y . \nabla \eta dx = \int_D u \eta dx \qquad \forall \eta \in \mathcal{V}. \tag{2.2}$$

We recall that $\mathcal{V}$ is continuously imbedded in $L^q(D)$ for all $q \in (1, +\infty)$. Consequently, Problem (2.2) admits a unique solution $y_u \in \mathcal{V}$ for any $u \in L^p(D)$, $p \in (1, +\infty)$. This allows to define the reduced cost functional

$$j : u \in L^p(D) \mapsto \frac{1}{2}\int_D (y_u - y^\dagger)^2 dx + \nu \int_D u dx.$$

Due to the non-convexity of the admissible set $\mathcal{E}$, the question of existence and uniqueness of a global minimizer of (2.1)-(2.2) is by no means trivial. We shall not discuss this issue here.

3. Optimality conditions

We begin by recalling a standard differentiability result. We give a quick proof for completeness.

Proposition 3.1. *Let $p \in (1, +\infty)$. The map*

$$\mathcal{J} : u \in L^p(D) \mapsto \frac{1}{2}\int_D (y_u - y^\dagger)^2 dx$$

is of class $\mathcal{C}^\infty$ in the sense of Fréchet. Its first order differential is given by

$$D\mathcal{J}(u)\delta = -\int_D z_u \delta dx \qquad \forall \delta \in L^p(D),$$

where the adjoint state $z_u \in \mathcal{V}$ solves

$$\int_D \nabla \eta . \nabla z_u dx = -\int_D (y_u - y^\dagger)\eta dx \qquad \forall \eta \in \mathcal{V}. \tag{3.1}$$

Proof. Set $a(y,z) = \int_D \nabla y . \nabla z dx$ and $\ell(u,z) = \int_D uz dx$. By virtue of the Sobolev imbedding recalled above, the map $(y,u) \in \mathcal{V} \times L^p(D) \mapsto a(y,.) - \ell(u,.) \in \mathcal{V}'$ is linear and continuous, and thus of class $\mathcal{C}^\infty$. In view of the coercivity of a, the implicit function theorem implies that the solution map $S : u \in L^p(D) \mapsto y_u \in \mathcal{V}$ is also of class $\mathcal{C}^\infty$. Then $\mathcal{J} : L^p(D) \to \mathbb{R}$ is of class $\mathcal{C}^\infty$ by composition. For any $\delta \in L^p(D)$, we obtain by the chain rule

$$D\mathcal{J}(u)\delta = \int_D (y_u - y^\dagger)(DS(u)\delta) dx.$$

Next, the adjoint equation (3.1) yields

$$D\mathcal{J}(u)\delta = -a(DS(u)\delta, z_u).$$

Besides, we have $a(S(u), z) = \ell(u,z)$ for all $(u,z) \in L^p(D) \times \mathcal{V}$, which, by differentiating with respect to u, provides

$$a(DS(u)\delta, z) = \ell(\delta, z) \qquad \forall (u, \delta, z) \in L^p(D) \times L^p(D) \times \mathcal{V}.$$

Thus we arrive at

$$D\mathcal{J}(u)\delta = -\ell(\delta, z_u).$$

□

Corollary 3.2. *Let $p \in (1, +\infty)$ be given. For all $u, v \in L^p(D)$, we have*

$$j(v) - j(u) = \int_D g_u(v-u)dx + R(u,v), \tag{3.2}$$

where the sensitivity g_u is defined by

$$g_u = -z_u + \nu,$$

and the remainder $R(u,v)$ enjoys the following properties.

(1) *For all $u, v \in L^p(D)$,*

$$R(u,v) \geq 0. \tag{3.3}$$

(2) *For all bounded subset A of $L^p(D)$, there exists $M > 0$ such that*

$$u, v \in A \Rightarrow R(u,v) \leq M\|v-u\|^2_{L^p(D)}. \tag{3.4}$$

Moreover, if $v = u + \beta\chi_{B(x_0,\rho)}$, $\beta \in \mathbb{R}$, $x_0 \in D$, $\rho > 0$, then, for all $\gamma \in (0,1)$, there holds

$$j(v) - j(u) = \pi\rho^2\beta g_u(x_0) + O(\rho^{\min(2+\gamma, 4/p)}). \tag{3.5}$$

Proof. The nonnegativity of $R(u,v)$ stems from the convexity of the cost functional J together with the linearity of the solution map S. One can also directly check that $R(u,v) = \frac{1}{2}\int_D (y_v - y_u)^2 dx$. The estimate (3.4) is a direct application of the Taylor-Lagrange inequality. If $v = u + \beta\chi_{B(x_0,\rho)}$, then (3.2)-(3.4) yield $j(v) - j(u) = \beta\int_{B(x_0,\rho)} g_u dx + O(\rho^{4/p})$. By elliptic regularity, g_u is locally H^2, thus γ-Hölder-continuous in the vicinity of x_0 for all $\gamma \in (0,1)$. This straightforwardly leads to (3.5). ☐

An expression of the form (3.5) is generally called *topological asymptotic expansion* (see, *e.g.*, [4, 12, 15, 21]). The function g_u is known as the *topological derivative* of the functional j with respect to the considered perturbation.

Definition 3.3. We say that a function $u \in \mathcal{E}$ is a local minimizer of (2.1)-(2.2) if

$$\exists \alpha > 0 \mid \forall v \in \mathcal{E},\ \|v-u\|_{L^1(D)} \leq \alpha \Rightarrow j(v) \geq j(u).$$

We say that it is a global minimizer if $j(v) \geq j(u)$ for all $v \in \mathcal{E}$.

Given a function $f : D \to \mathbb{R}$ and a number $a \in \mathbb{R}$, we denote $[f = a] := \{x \in D, f(x) = a\}$. If A is a subset of D we denote by $\mathrm{int}(A)$ its interior. In view of Corollary 3.2 and Definition 3.3 we deduce the following proposition.

Proposition 3.4. *A necessary condition for $u \in \mathcal{E}$ to be a local minimizer of* (2.1)-(2.2) *is*

$$\begin{cases} g_u \geq 0 \ \text{in } \mathrm{int}([u = u^+]), \\ g_u \leq 0 \ \text{in } \mathrm{int}([u = u^-]). \end{cases} \tag{3.6}$$

A sufficient condition for $u \in \mathcal{E}$ to be a global minimizer of (2.1)-(2.2) *is*

$$\begin{cases} g_u \geq 0 \ \text{a.e. in } [u = u^+], \\ g_u \leq 0 \ \text{a.e. in } [u = u^-]. \end{cases} \tag{3.7}$$

Proof. The necessary condition (3.6) is a consequence of the expansion (3.5), where γ and p are chosen arbitrarily in $(0,1)$ and $(1,2)$, respectively. To prove that the condition (3.7) is sufficient, we consider another function $v \in \mathcal{E}$. The variation $v - u$ can be expressed as

$$v - u = (u^+ - u^-)\chi^- + (u^- - u^+)\chi^+,$$

where χ^- is the characteristic function of some subset ω^- of $[u = u^-]$ and χ^+ is the characteristic function of some subset ω^+ of $[u = u^+]$. Using (3.2) and (3.3) it comes

$$j(v) - j(u) \geq \int_{\omega^-} g_u(u^+ - u^-)dx + \int_{\omega^+} g_u(u^- - u^+)dx.$$

Both integrals are nonnegative, which completes the proof. ☐

4. Description of the algorithm

To each continuous function $\psi : D \to \mathbb{R}$ we associate the function $u_\psi \in \mathcal{E}$ defined by

$$u_\psi(x) = \begin{cases} u^+ & \text{if } \psi(x) \geq 0, \\ u^- & \text{if } \psi(x) < 0. \end{cases} \tag{4.1}$$

In the sequel we will use the simplified notations

$$j(\psi) := j(u_\psi), \qquad g_\psi := g_{u_\psi}.$$

We will denote by $\mathcal{L}^n$ and $\mathcal{H}^n$ the n-dimensional Lebesgue and Hausdorff measures, respectively. The first step in the construction of the algorithm is to reformulate Proposition 3.4 in the level set framework.

Lemma 4.1. *Let $\psi \in \mathcal{C}(D)$ be such that $\mathcal{L}^2([\psi = 0]) = 0$. Then u_ψ is a global minimizer of* (2.1)-(2.2) *if and only if, for all $x \in D$, there holds*

$$\begin{cases} \psi(x) > 0 & \Rightarrow \quad g_\psi(x) \geq 0, \\ \psi(x) < 0 & \Rightarrow \quad g_\psi(x) \leq 0. \end{cases} \tag{4.2}$$

Proof. By continuity of ψ, we have $\psi(x) > 0 \Rightarrow x \in \operatorname{int}([u_\psi = u^+])$ and $\psi(x) < 0 \Rightarrow x \in \operatorname{int}([u_\psi = u^-])$. Then using (3.6) yields directly the necessary condition. Let us now derive the sufficient condition from (3.7). Thus, let x belong to $[u_\psi = u^+] \cap [g_\psi < 0]$. It stems from (4.2) that $\psi(x) \leq 0$, hence $x \in [\psi = 0]$ or $x \in [\psi < 0] \subset \operatorname{int}([u_\psi = u^-])$. The second situation being impossible, it comes $[u_\psi = u^+] \cap [g_\psi < 0] \subset [\psi = 0]$, which is of zero Lebesgue measure. Likewise, $\mathcal{L}^2([u_\psi = u^-] \cap [g_\psi > 0]) = 0$, and the proof is complete. $\square$

With the assumptions made, we have that $g_\psi \in H^{3/2}(D)$ (see, *e.g.*, [20]), which is imbedded in $\mathcal{C}^{0,1/2}(\bar{D})$. This regularity is however not sufficient to use g_ψ as a descent direction. We will introduce an auxiliary function $\tilde{g}_\psi$ with higher regularity and the relevant properties.

For any $\sigma \geq 0$, we denote by $\langle ., . \rangle_\sigma$ the canonical scalar product on the Sobolev space $H^\sigma(D)$ and by $\|.\|_\sigma$ the associated norm. We denote by $\mathcal{S}_\sigma$ the unit ball of $H^\sigma(D)$, *i.e.*,

$$\mathcal{S}_\sigma = \{\psi \in H^\sigma(D), \|\psi\|_\sigma = 1\}.$$

We make the following assumption.

Assumption 4.2. There exists $\sigma > 2$ such that the following holds.

(1) For all u in $\mathcal{E}$, there exists a function $\tilde{g}_u \in \mathcal{S}_\sigma$ which satisfies, for all $x \in D$,

$$\tilde{g}_u(x) > 0 \Leftrightarrow g_u(x) > 0,$$
$$\tilde{g}_u(x) < 0 \Leftrightarrow g_u(x) < 0,$$

(2) The map $u \in \mathcal{E} \mapsto \tilde{g}_u$ is continuous for the norms $\|.\|_{L^1(D)} \to \|.\|_\sigma$.

Note that, when D is smooth, $\mathcal{V} = H^1_0(D)$, $y^\dagger \in H^s(D)$, $s > 0$, and $\nu \neq 0$, we can simply take $\tilde{g}_u = g_u / \|g_u\|_\sigma$. Indeed we have by elliptic regularity that $g_u \in H^{\min(2,s)+2}(D)$ for all u in $\mathcal{E}$; furthermore, $g_u \neq 0$ since $g_{u|\partial D} = \nu \neq 0$.

We now fix σ so that Assumption 4.2 is fulfilled. The proposed algorithm is based on the observation that the equivalence relation

$$\exists \lambda > 0 \mid \tilde{g}_\psi = \lambda \psi \tag{4.3}$$

is a sufficient optimality condition for any function $\psi \in H^\sigma(D)$ which is nonzero almost everywhere (see Lemma 4.1). Of course, there holds

$$u_{\lambda\psi} = u_\psi \qquad \forall \psi \in H^\sigma(D),\ \forall \lambda > 0, \tag{4.4}$$

thus, without any loss of generality, we will restrict the search to unitary functions ψ. More precisely, we will apply the fixed point iteration on the unit sphere $\mathcal{S}_\sigma$ to the solution of (4.3), which becomes

$$\tilde{g}_\psi = \psi. \tag{4.5}$$

To do so, for all $\psi, \varphi \in \mathcal{S}_\sigma$ and all $\kappa \in [0, 1]$, we denote by θ the non-oriented angle between ψ and φ, that is

$$\theta = \arccos \langle \psi, \varphi \rangle_\sigma. \tag{4.6}$$

If the vectors ψ and φ are linearly independent, we denote by $C_\kappa(\psi,\varphi)$ the image of ψ by the rotation of angle $\kappa\theta$ in the oriented plane generated by the pair (ψ,φ). We have by this definition

$$C_\kappa(\psi,\varphi) = \cos(\kappa\theta)\psi + \sin(\kappa\theta)\frac{k}{\|k\|},$$

where k denotes the orthogonal projection of φ onto the orthogonal hyperplane $\psi^\perp$ of ψ, *i.e.*, $k = \varphi - \cos\theta\psi$. A straightforward calculation results in

$$C_\kappa(\psi,\varphi) = \frac{1}{\sin\theta}\left[\sin((1-\kappa)\theta)\psi + \sin(\kappa\theta)\varphi\right].$$

If $\psi = \pm\varphi$, we set by convention $C_\kappa(\psi,\varphi) = \varphi$ for all $\kappa \in [0,1]$. The structure of the algorithm is the following.

Algorithm

(1) Set $n = 0$ and choose some $\psi_0 \in \mathcal{S}_\sigma$.
(2) If (4.2) is satisfied then stop else set

$$\psi_{n+1} = C_{\kappa_n}(\psi_n, \tilde{g}_{\psi_n})$$

with some $\kappa_n \in [0,1]$.
(3) Increment $n \leftarrow n+1$ and goto (2).

At each iteration, the step $\kappa_n \in [0,1]$ is fixed by a line search. This issue will be discussed in Section 8.

5. Variation of the objective functional with respect to boundary perturbations

We define the set

$$\mathcal{G} = \{\psi \in H^\sigma(D), [\psi = 0] \subset\subset D, \nabla\psi \neq 0 \text{ on } [\psi = 0]\}.$$

We recall that $\sigma > 2$, hence $H^\sigma(D)$ is continuously imbedded in $\mathcal{C}^1(\bar{D})$ (see, *e.g.*, [10]). Hence, for all $\psi \in \mathcal{G}$, the level set $[\psi = 0]$ is compact, and thus, $\inf_{[\psi=0]} |\nabla\psi| > 0$. We begin by a continuity result.

Lemma 5.1. *Let $\psi \in \mathcal{G}$. There exists $c_1, c_2 > 0$ such that for all $\varphi \in L^\infty(D)$,*

$$\|\varphi\|_{L^\infty(D)} \leq c_1 \Rightarrow \|u_{\psi+\varphi} - u_\psi\|_{L^1(D)} \leq c_2\|\varphi\|_{L^\infty(D)}.$$

Proof. Let $\psi \in \mathcal{G}$ and $\varphi \in L^\infty(D)$. We have

$$\|u_{\psi+\varphi} - u_\psi\|_{L^1(D)} = (u^- - u^+)\left[\int_{[-\varphi\leq\psi<0]} dx + \int_{[0\leq\psi<-\varphi]} dx\right].$$

Hence

$$\|u_{\psi+\varphi} - u_\psi\|_{L^1(D)} \leq (u^- - u^+)\int_{[-\|\varphi\|_{L^\infty(D)}\leq\psi<\|\varphi\|_{L^\infty(D)}]} dx.$$

We can rewrite this latter integral as

$$\int_{[-\|\varphi\|_{L^\infty(D)}\leq\psi<\|\varphi\|_{L^\infty(D)}]} dx = F(\|\varphi\|_{L^\infty(D)}) - F(-\|\varphi\|_{L^\infty(D)}),$$

with

$$F(t) = \int_{[\psi<t]} dx.$$

By Corollary A.5, the function F is differentiable at 0, which enables to conclude. $\square$

Lemma 5.2. *For all $\psi \in \mathcal{G}$ and $\varphi \in \mathcal{C}^1(\bar{D})$, we have*

$$j(\psi + t\varphi) - j(\psi) = -t(u^- - u^+)\int_{[\psi=0]} \frac{g_\psi\varphi}{|\nabla\psi|} d\mathcal{H}^1 + o(t).$$

Proof. Corollary 3.2 entails, for any $p > 1$,

$$j(\psi + t\varphi) - j(\psi) = \int_D g_\psi (u_{\psi+t\varphi} - u_\psi) dx + O(\|u_{\psi+t\varphi} - u_\psi\|^2_{L^p(D)}).$$

By definition of u_ψ, this expression can be rewritten as

$$j(\psi + t\varphi) - j(\psi) = u^+ \left[\int_{[\psi+t\varphi \geq 0]} g_\psi dx - \int_{[\psi \geq 0]} g_\psi dx \right] + u^- \left[\int_{[\psi+t\varphi < 0]} g_\psi dx - \int_{[\psi < 0]} g_\psi dx \right] + O(\|u_{\psi+t\varphi} - u_\psi\|^{2/p}_{L^1(D)}).$$

By using Corollary A.5 and Lemma 5.1, it comes for t small enough

$$j(\psi + t\varphi) - j(\psi) = u^+ t \int_{[\psi=0]} \frac{g_\psi \varphi}{|\nabla \psi|} d\mathcal{H}^1 - u^- t \int_{[\psi=0]} \frac{g_\psi \varphi}{|\nabla \psi|} d\mathcal{H}^1 + o(t) + O(t^{2/p}).$$

Choosing $p \in (1, 2)$ completes the proof. □

Lemma 5.3. *Let $\psi \in \mathcal{G}$. We assume that u_ψ is a local minimizer of* (2.1)-(2.2). *Then for all $x \in D$*

$$\psi(x) = 0 \quad \Rightarrow \quad g_\psi(x) = 0. \tag{5.1}$$

Proof. It suffices to take $\varphi = \tilde{g}_\psi$ in Lemma 5.2. Another proof consists in observing that $[\psi = 0] \subset \overline{[\psi < 0]} \cap \overline{[\psi > 0]}$, due to the non-vanishing of $\nabla\psi$ on $[\psi = 0]$, and using (4.2) together with the continuity of g_ψ. □

Lemma 5.4. *Let $\psi \in \mathcal{S}_\sigma \cap \mathcal{G}$, $\varphi \in \mathcal{S}_\sigma$ be two linearly independent functions. We have*

$$j(C_\kappa(\psi, \varphi)) - j(\psi) = -\kappa \frac{\theta}{\sin\theta} (u^- - u^+) \int_{[\psi=0]} \frac{g_\psi \varphi}{|\nabla \psi|} d\mathcal{H}^1 + o(\kappa). \tag{5.2}$$

Proof. On the one hand, we observe that, for all $\kappa \in (0, 1)$,

$$C_\kappa(\psi, \varphi) = \frac{\sin((1-\kappa)\theta)}{\sin\theta} \left[\psi + \frac{\sin(\kappa\theta)}{\sin((1-\kappa)\theta)} \varphi \right], \tag{5.3}$$

which, by virtue of (4.4), yields

$$j(C_\kappa(\psi, \varphi)) - j(\psi) = j(\psi + \frac{\sin(\kappa\theta)}{\sin((1-\kappa)\theta)} \varphi) - j(\psi). \tag{5.4}$$

On the other hand, a straightforward Taylor expansion provides

$$\frac{\sin(\kappa\theta)}{\sin((1-\kappa)\theta)} = \frac{\kappa\theta}{\sin\theta} + o(\kappa).$$

Applying Lemma 5.2 completes the proof. □

Lemma 5.5. *Let $\psi \in \mathcal{S}_\sigma \cap \mathcal{G}$ be such that $\mathcal{H}^1([\psi = 0]) > 0$. We assume that the condition* (5.1) *is not satisfied. Then there exists $\bar{\kappa} > 0$ such that, for all $\kappa \in (0, \bar{\kappa}]$, $C_\kappa(\psi, \tilde{g}_\psi) \in \mathcal{G}$ and*

$$j(C_\kappa(\psi, \tilde{g}_\psi)) < j(\psi).$$

Proof. We first remark that, in view of the assumptions, ψ and $\tilde{g}_\psi$ are linearly independent. Then, by virtue of Lemma 5.4 we have

$$j(C_\kappa(\psi, \tilde{g}_\psi)) - j(\psi) = -\kappa \frac{\theta}{\sin\theta} (u^- - u^+) \int_{[\psi=0]} \frac{g_\psi \tilde{g}_\psi}{|\nabla \psi|} d\mathcal{H}^1 + o(\kappa).$$

This quantity is negative provided that κ is small enough. The fact that $C_\kappa(\psi, \tilde{g}_\psi) \in \mathcal{G}$ for κ small is a direct consequence of the regularity of the functions ψ and $\tilde{g}_\psi$. □

6. Variation of the objective functional with respect to topological perturbations

In this section we study the variation of the objective functional when the level set function ψ is perturbed in such a way that the existing connected components of the interface $[\psi = 0]$ are preserved, while new ones are created. This corresponds to the case where (5.1) is satisfied. Then the leading term in the asymptotic expansion (5.2) vanishes, thus higher order terms must be considered. We start by two preliminary lemmas.

Lemma 6.1. *For all $\psi, \varphi \in H^\sigma(D)$, for all $t \geq 0$, we have*

$$\varphi\,[u_{\psi+t\varphi} - u_\psi] \leq 0 \text{ in } D.$$

Proof. Let $x \in D$ be such that $u_{\psi+t\varphi}(x) - u_\psi(x) < 0$. This means that $u_{\psi+t\varphi}(x) = u^+$ and $u_\psi(x) = u^-$. Thus $(\psi + t\varphi)(x) \geq 0$ and $\psi(x) < 0$. Hence $\varphi(x) > 0$. We obtain similarly that $u_{\psi+t\varphi}(x) - u_\psi(x) > 0 \Rightarrow \varphi(x) < 0$. $\square$

Lemma 6.2. *Let $\psi \in \mathcal{S}_\sigma \cap \mathcal{G}$ be such that* (5.1) *is satisfied. Then the function*

$$\zeta_\psi := -\frac{\tilde{g}_\psi}{\psi}$$

extends by continuity on $[\psi = 0]$.

Proof. We study the continuity of ζ_ψ at some point $\hat{x} \in [\psi = 0]$. Let $x \in D$ be in a neighborhood of $\hat{x}$. We denote by x_h a projection of x onto the compact set $[\psi = 0]$. Due to the $\mathcal{C}^1$ regularity of the curve $[\psi = 0]$, this projection is orthogonal, *i.e.*, the vector $x - x_h$ is colinear to the normal n_h at x_h. By a Taylor expansion, we obtain

$$\psi(x) = \psi(x_h) + \nabla\psi(x_h).(x - x_h) + o(|x - x_h|).$$

Since ψ vanishes on $[\psi = 0]$, it comes

$$\psi(x) = \partial_n\psi(x_h)(x - x_h).n_h + o(|x - x_h|).$$

Similarly, we have

$$\tilde{g}_\psi(x) = \partial_n\tilde{g}_\psi(x_h)(x - x_h).n_h + o(|x - x_h|).$$

By assumption, $\partial_n\psi$ does not vanish on $[\psi = 0]$. In addition, $|x - x_h| \leq |x - \hat{x}|$ and $|x_h - \hat{x}| \leq |x_h - x| + |x - \hat{x}| \leq 2|x - \hat{x}|$. Hence

$$\zeta_\psi(x) = -\frac{\partial_n\tilde{g}_\psi(x_h) + o(1)}{\partial_n\psi(x_h) + o(1)} \to -\frac{\partial_n\tilde{g}_\psi(\hat{x})}{\partial_n\psi(\hat{x})} \qquad (x \to \hat{x}).$$

$\square$

We define the set $\mathcal{T}$ as the set of all functions $\psi \in \mathcal{G}$ satisfying (5.1) as well as the following conditions:

- either (4.2) is fulfilled,
- or $\operatorname{argmax}\{\zeta_\psi(x), x \in \bar{D}\} = \{x^\star\}$ with $\psi(x^\star) \neq 0$, and there exists $\alpha > 0$ such that $\nabla\zeta_\psi(x).(x - x^\star) \leq -\alpha|x - x^\star|^2$ for all x in a neighborhood of $x^\star$ and such that $\psi(x) \neq 0$.

The above condition roughly asserts that ζ_ψ is not flat near its maximum. The proposed formulation enables simple proofs, but it is not sharp.

Lemma 6.3. *Let $\psi \in \mathcal{S}_\sigma \cap \mathcal{T}$. We assume that* (4.2) *is not satisfied. Then there exist two positive constants $\underline{t} < \bar{t}$ such that*

$$j(\psi + t\tilde{g}_\psi) < j(\psi) \qquad \forall t \in (\underline{t}, \bar{t}). \tag{6.1}$$

Moreover, $\psi + t\tilde{g}_\psi \in \mathcal{G}$ for all $t \in (\underline{t}, \bar{t})$.

Proof. (1) First, we note that, as (4.2) is not satisfied, we have by assumption $\operatorname{argmax}\{\zeta_\psi(x), x \in \bar{D}\} = \{x^\star\}$ with $\psi(x^\star) \neq 0$, and there exists $\alpha > 0$ such that $\nabla\zeta_\psi(x).(x - x^\star) \leq -\alpha|x - x^\star|^2$ in the vicinity of $x^\star$. For all $t > 0$ we set

$$\Delta_t = [\psi \geq 0 \ \wedge \ \psi + t\tilde{g}_\psi < 0] \cup [\psi < 0 \ \wedge \ \psi + t\tilde{g}_\psi \geq 0].$$

In fact, in view of (5.1), there holds $\psi(x) \neq 0$ for all $x \in \Delta_t$. This permits to obtain the inclusions

$$[\psi \neq 0 \ \wedge \ \zeta_\psi > \frac{1}{t}] \subset \Delta_t \subset [\zeta_\psi \geq \frac{1}{t}]. \tag{6.2}$$

It stems from the assumptions that, in the vicinity of $x^\star$,

$$\zeta_\psi(x) - \zeta_\psi(x^\star) \le -\frac{\alpha}{2}|x - x^\star|^2.$$

Therefore there exists $\bar{t} > \zeta_\psi(x^\star)^{-1}$ such that

$$\forall t < \bar{t},\ x \in \Delta_t \Rightarrow \zeta_\psi(x^\star) - \frac{\alpha}{2}|x - x^\star|^2 \ge \frac{1}{t}. \tag{6.3}$$

We now set $\underline{t} = \zeta_\psi(x^\star)^{-1}$ and, for all $t > \underline{t}$,

$$\rho(t) = \sqrt{\frac{2}{\alpha}\left(\frac{1}{\underline{t}} - \frac{1}{t}\right)}.$$

Then we immediately derive from (6.3) that

$$\Delta_t \subset \bar{B}(x^\star, \rho(t)) \qquad \forall t \in (\underline{t}, \bar{t}). \tag{6.4}$$

By Corollary 3.2 we have for any $p > 1$

$$j(\psi + t\tilde{g}_\psi) - j(\psi) = \int_D g_\psi (u_{\psi + t\tilde{g}_\psi} - u_\psi) dx + O(\|u_{\psi + t\tilde{g}_\psi} - u_\psi\|^2_{L^p(D)}).$$

By Lemma 6.1 we can write

$$j(\psi + t\tilde{g}_\psi) - j(\psi) = -(u^- - u^+) \int_{\Delta_t} |g_\psi| dx + O(\mathcal{L}^2(\Delta_t)^{2/p}).$$

Yet, in view of (6.4), the continuity of g_ψ and the fact that $g_\psi(x^\star) \neq 0$, we have $|g_\psi(x)| \ge m > 0$ for all $x \in \Delta_t$ provided that t is sufficiently small. Possibly decreasing $\bar{t}$ in this respect, we get

$$j(\psi + t\tilde{g}_\psi) - j(\psi) \le -(u^- - u^+) m \mathcal{L}^2(\Delta_t) + O(\mathcal{L}^2(\Delta_t)^{2/p}) \qquad \forall t \in (\underline{t}, \bar{t}).$$

In addition, for $t > \underline{t}$, Δ_t has nonzero Lebesgue measure by virtue of (6.2) and the continuity of ζ_ψ. We deduce (6.1) by possibly decreasing $\bar{t}$ again and choosing some $p \in (1, 2)$.

(2) We now check that $\psi + t\tilde{g}_\psi \in \mathcal{G}$ for t suitably chosen, namely, that $\nabla(\psi + t\tilde{g}_\psi)$ does not vanish on $[\psi + t\tilde{g}_\psi = 0]$. Let $\hat{x} \in D$ and $t > \underline{t}$ be such that $\psi(\hat{x}) + t\tilde{g}_\psi(\hat{x}) = 0$. We distinguish between two cases, depending on whether $\psi(\hat{x}) = 0$ or $\psi(\hat{x}) \neq 0$.

- We examine first the case where $\psi(\hat{x}) \neq 0$. Then a simple calculation results in

  $$\nabla(\psi + t\tilde{g}_\psi)(\hat{x}) = -t\psi(\hat{x})\nabla\zeta_\psi(\hat{x}).$$

  We have $\zeta(\hat{x}) = \frac{1}{t}$, hence $|\hat{x} - x^\star| \le \rho(t)$. Thus there exists $\beta > 0$ (independent of $\hat{x}$) such that $|\zeta_\psi(\hat{x}) - \zeta_\psi(x^\star)| \le \beta|\hat{x} - x^\star|$ provided that t is sufficiently small, since ζ_ψ is $\mathcal{C}^1$ in the vicinity of $x^\star$ (possibly decrease $\bar{t}$). In addition, we have by assumption $|\nabla\zeta(\hat{x})| \ge \alpha|\hat{x} - x^\star|$. This implies

  $$|\nabla(\psi + t\tilde{g}_\psi)(\hat{x})| \ge \frac{\alpha}{\beta} t|\psi(\hat{x})||\zeta_\psi(\hat{x}) - \zeta_\psi(x^\star)| = \frac{\alpha}{\beta} t^2 |\tilde{g}_\psi(\hat{x})| \left|\frac{1}{t} - \frac{1}{\underline{t}}\right|.$$

  By continuity, since $\tilde{g}(x^\star) \neq 0$, we have $|\tilde{g}_\psi(\hat{x})| \ge m > 0$, with m independent of $\hat{x}$. We deduce that

  $$|\nabla(\psi + t\tilde{g}_\psi)(\hat{x})| \ge m\frac{\alpha}{\beta}\frac{t}{\underline{t}}(t - \underline{t}).$$

- Consider now a point $\hat{x}$ such that $\psi(\hat{x}) = \tilde{g}_\psi(\hat{x}) = 0$. There exists $\delta > 0$ and a neighborhood $\mathcal{O}$ of $[\psi = 0]$ such that

  $$\zeta_\psi(x) \le \zeta_\psi(x^\star) - \delta \ \forall x \in \mathcal{O}.$$

  We derive that, for all $x \in \mathcal{O} \cap [\psi > 0]$,

  $$\psi(x) + t\tilde{g}_\psi(x) = [1 - t\zeta_\psi(x)]\,\psi(x) \ge \left[1 - t\left(\frac{1}{\underline{t}} - \delta\right)\right]\psi(x).$$

  We impose $\delta < \frac{1}{\underline{t}}$ and we choose $k \in (0, \delta\underline{t})$. We assume that $t \in (\underline{t}, \underline{t}\frac{1-k}{1-\delta\underline{t}})$ (possibly decrease $\bar{t}$). It comes

  $$\psi(x) + t\tilde{g}_\psi(x) \ge k\psi(x) \qquad \forall x \in \mathcal{O} \cap [\psi > 0].$$

This inequality can be rewritten as

$$[\psi(x) + t\tilde{g}_\psi(x)] - [\psi(\hat{x}) + t\tilde{g}_\psi(\hat{x})] \geq k[\psi(x) - \psi(\hat{x})] \geq 0 \qquad \forall x \in \mathcal{O} \cap [\psi > 0].$$

We choose $x = \hat{x} + \lambda n$ with $\lambda > 0$ and n the unit normal to $[\psi = 0]$ oriented towards $[\psi > 0]$. By letting λ go to zero it comes

$$\partial_n[\psi + t\tilde{g}_\psi](\hat{x}) \geq k\partial_n\psi(\hat{x}) \geq 0.$$

Thus,

$$|\nabla(\psi + t\tilde{g}_\psi)(\hat{x})| \geq k|\partial_n\psi(\hat{x})| > 0.$$

□

Lemma 6.4. *Under the assumptions of Lemma 6.3, there exist two positive numbers $\underline{\kappa} < \overline{\kappa}$ such that, for all $\kappa \in (\underline{\kappa}, \overline{\kappa})$, $C_\kappa(\psi, \tilde{g}_\psi) \in \mathcal{G}$ and*

$$j(C_\kappa(\psi, \tilde{g}_\psi)) < j(\psi). \tag{6.5}$$

Proof. From (5.4), we get

$$j(C_\kappa(\psi, \tilde{g}_\psi)) - j(\psi) = j(\psi + t_\kappa\tilde{g}_\psi) - j(\psi)$$

with

$$t_\kappa = \frac{\sin(\kappa\theta)}{\sin((1-\kappa)\theta)}.$$

We easily check that the map $\kappa \mapsto t_\kappa$ is an increasing homeomorphism of $(0,1)$ into $(0,+\infty)$. Applying Lemma 6.3 leads to (6.5). We prove that $C_\kappa(\psi, \tilde{g}_\psi) \in \mathcal{G}$ by using the factorization (5.3) together with the fact that $\psi + t_\kappa\tilde{g}_\psi \in \mathcal{G}$. □

7. Convergence of the algorithm

Let $\tilde{j}$ be the lower semi-continuous regularization of the function $\psi \in H^\sigma(D) \mapsto j(\psi)$, *i.e.*,

$$\tilde{j}(\psi) = \liminf_{\varphi \to \psi} j(\varphi).$$

Note that, at all point $\psi \in \mathcal{G}$, j is continuous by virtue of Lemma 5.1, thus $\tilde{j}(\psi) = j(\psi)$. We now assume that the step κ_n at iteration n of the algorithm is chosen such that

$$\tilde{j}(C_{\kappa_n}(\psi_n, \tilde{g}_{\psi_n})) = \min_{\kappa \in [0,1]} \tilde{j}(C_\kappa(\psi_n, \tilde{g}_{\psi_n})).$$

The following theorem states that if the algorithm converges, or at least it has accumulation points, then the resulting domains are optimal, under some conditions. Because of the non-compactness of $\mathcal{S}_\sigma$, the existence of accumulation points cannot be proved in general in the infinite dimensional setting. However, after discretization of $\mathcal{S}_\sigma$, accumulation points will always exist, and the numerical computations shown in Section 9 seem to produce mesh independent solutions.

Theorem 7.1. *Let $\psi^\star \in \mathcal{S}_\sigma$ be an accumulation point of the sequence (ψ_n) generated by the algorithm. We assume that $\mathcal{H}^1([\psi = 0]) > 0$.*

(1) *If $\psi^\star \in \mathcal{G}$, then $\psi^\star$ satisfies the geometrical optimality condition* (5.1).
(2) *If $\psi^\star \in \mathcal{T}$, then $\psi^\star$ satisfies the topological optimality condition* (4.2).

Proof. (1) We argue by contradiction by assuming that $\psi^\star$ is an accumulation point which does not satisfy (5.1). By Lemma 5.5, there exists $\kappa > 0$ such that $C_\kappa(\psi^\star, \tilde{g}_{\psi^\star}) \in \mathcal{G}$ and

$$\tilde{j}(C_\kappa(\psi^\star, \tilde{g}_{\psi^\star})) < \tilde{j}(\psi^\star). \tag{7.1}$$

Consider a subsequence, still denoted by $(\psi_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to+\infty} \|\psi_n - \psi^\star\|_\sigma = 0$. By Lemma 5.1, we also have $\lim_{n\to+\infty} \|u_{\psi_n} - u_{\psi^\star}\|_{L^1(D)} = 0$, thus $\lim_{n\to+\infty} \|\tilde{g}_{\psi_n} - \tilde{g}_{\psi^\star}\|_\sigma = 0$. This clearly entails $\lim_{n\to+\infty} \|C_\kappa(\psi_n, \tilde{g}_{\psi_n}) - C_\kappa(\psi^\star, \tilde{g}_{\psi^\star})\|_\sigma = 0$. Hence by continuity $\lim_{n\to+\infty} j(C_\kappa(\psi_n, \tilde{g}_{\psi_n})) = j(C_\kappa(\psi^\star, \tilde{g}_{\psi^\star})) = \tilde{j}(C_\kappa(\psi^\star, \tilde{g}_{\psi^\star}))$. Therefore, in view of (7.1), there exists $N \in \mathbb{N}$ such that

$$\tilde{j}(C_\kappa(\psi_n, \tilde{g}_{\psi_n})) < \tilde{j}(\psi^\star) \qquad \forall n \geq N. \tag{7.2}$$

Next, as the sequence $(\tilde{j}(\psi_n))_{n\in\mathbb{N}}$ is non-increasing, we have

$$\tilde{j}(\psi^\star) \leq \tilde{j}(\psi_{n+1}) = \tilde{j}(C_{\kappa_n}(\psi_n, \tilde{g}_{\psi_n})) \qquad \forall n \in \mathbb{N}.$$

Using now the optimality of the step, it comes

$$\tilde{j}(\psi^\star) \leq \tilde{j}(C_\kappa(\psi_n, \tilde{g}_{\psi_n})) \qquad \forall n \in \mathbb{N}. \tag{7.3}$$

Comparing (7.2) and (7.3) leads to a contradiction.

(2) The same argument applies in the second case, based on Lemma 6.4.

□

Remark 7.2. (1) The convexity of J has been used only to prove that the necessary optimality condition (4.2) is also sufficient.

(2) In case the algorithm converges to a function $\psi^\star$ which does not belong to $\mathcal{G}$, a reinitialization of $\psi^\star$ may be performed and the iterations continued (see, *e.g.*, [3]). If $\psi^\star \in \mathcal{G} \setminus \mathcal{T}$, then either $\psi^\star$ or $\tilde{g}_{\psi^\star}$ may be reconstructed.

(3) The lower semi-continuous regularization of the objective functional has been introduced for theoretical purposes only, in order to guarantee the existence of an optimal step.

8. Numerical algorithm

8.1. **Discretization of the state equation and discrete sensitivity.** We consider a finite element approximation of the state equation (2.2) with Lipschitz continuous basis functions. Given a finite dimensional subspace $\mathcal{V}^h$ of $\mathcal{V}$, this leads to the semi-discrete optimization problem

$$\underset{(u,y)\in\mathcal{E}\times\mathcal{V}^h}{\text{Minimize}} \; J(u,y) = \frac{1}{2}\int_D (y-y^\dagger)^2 dx + \nu \int_D u dx \tag{8.1}$$

subject to

$$\int_D \nabla y . \nabla \eta dx = \int_D u \eta dx \qquad \forall \eta \in \mathcal{V}^h. \tag{8.2}$$

For any $u \in \mathcal{E}$ and $y_u^h \in \mathcal{V}^h$ solution of (8.2), we also define the discrete adjoint state $z_u^h \in \mathcal{V}^h$ as the solution of

$$\int_D \nabla \eta . \nabla z_u^h dx = -\int_D (y_u^h - y^\dagger)\eta dx \qquad \forall \eta \in \mathcal{V}^h, \tag{8.3}$$

and we set $g_u^h = -z_u^h + \nu$. Then one can check that Proposition 3.1, Corollary 3.2 and Proposition 3.4 remain valid by replacing y_u, z_u and g_u by their discrete counterparts. In fact, g_u^h is the exact sensitivity for the semi-discrete problem. Now, we associate to any continuous function $\psi : D \to \mathbb{R}$ a function u_ψ according to (4.1). Then Lemma 4.1 extends to the semi-discrete problem in the following way.

Lemma 8.1. *Let $\psi \in \mathcal{C}(D)$ be such that $\mathcal{L}^2([\psi = 0]) = 0$. Then u_ψ^h is a global minimizer of* (8.1)-(8.2) *if and only if, for all $x \in D$, there holds*

$$\begin{cases} \psi(x) > 0 & \Rightarrow \quad g_\psi^h(x) \geq 0, \\ \psi(x) < 0 & \Rightarrow \quad g_\psi^h(x) \leq 0. \end{cases} \tag{8.4}$$

Therefore, provided that $\mathcal{L}^2([g_u^h = 0]) = 0$ at the optimum, the optimal domain can be exactly represented by a level set function $\psi \in \mathcal{V}^h$ (take $\psi = g_u^h$). Thus we restrict the search to level set functions $\psi \in \mathcal{V}^h$, resulting in a full discretization of the problem. For computing θ in (4.6), we endow $\mathcal{V}^h$ with an arbitrary scalar product. The corresponding unit sphere of $\mathcal{V}^h$ is denoted by $\mathcal{S}^h$.

Numerical algorithm

(1) Set $n = 0$, choose $\psi_0 \in \mathcal{S}^h$ and $a > 0$.

(2) If $\mathcal{L}^2([\psi_n g_{\psi_n} < 0]) \leq a$ then stop else set

$$\psi_{n+1} = C_{\kappa_n}(\psi_n, \tilde{g}_{\psi_n}^h)$$

with some $\kappa_n \in [0,1]$ fixed by line search.

(3) Increment $n \leftarrow n+1$ and goto (2).

8.2. **Line search.** The initial step is chosen as $\kappa_0 = 1$. Then, at iteration $n \geq 1$, κ_n is fixed by the following procedure.

(1) Initialize $\kappa_n = \min(1, \kappa_{n-1} \times 1.5)$.
(2) Loop while $j(C_{\kappa_n}(\psi_n, \tilde{g}^h_{\psi_n})) > j(\psi_n)$

$$\kappa_n \leftarrow \kappa_n/2.$$

End loop.

9. Numerical examples

In the subsequent computations, the computational domain is $D = [-1, 1] \times [0, 1]$ and the different data and parameters are chosen as follows: $u^+ = 0$, $u^- = 1$, $a = 10^{-3}$, $\psi_0 = -1/\|1\|$. We use $P1$ finite elements with exact integration, and equip $\mathcal{V}^h$ with the L^2 scalar product. We take as descent direction the normalized sensitivity, *i.e.*, $\tilde{g}^h_u = g^h_u/\|g^h_u\|$, and we do not perform any reinitialization. The implementation is done under Matlab.

We present four computations, which differ by the value of ν, the function $y^\dagger$ and the mesh used (see Figures 1 through 4). The function $y^\dagger$ is chosen among:

$$y^{\dagger,1} = 0.05, \qquad y^{\dagger,2}(x_1, x_2) = 0.05 + 0.1 \sin(\pi x_1) \sin(\pi x_2).$$

Some numerical data are reported in Table 1. The value of θ indicated is the one obtained at convergence. The CPU time is measured on a PC with 2.4 GHz processor.

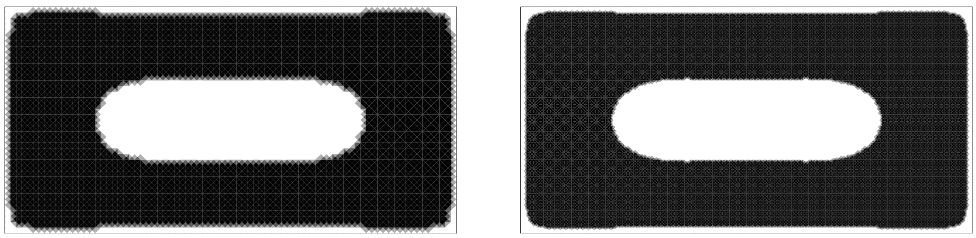

Figure 1. Obtained domains at convergence for $\nu = 10^{-4}$, $y^\dagger = y^{\dagger,1}$: coarse mesh (case 1, left) and fine mesh (case 2, right). The black region represents the set $[u = u^-]$.

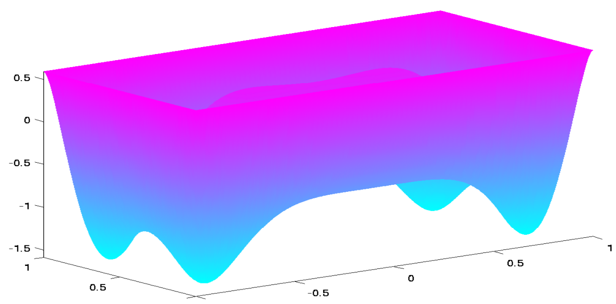

Figure 2. Level set function at convergence for case 2.

Case	ν	$y^\dagger$	Nb. of nodes	θ (degrees)	CPU time (s)
1	10^{-4}	$y^{\dagger,1}$	6521	0.027	18.5
2	10^{-4}	$y^{\dagger,1}$	25841	0.095	112.8
3	2×10^{-3}	$y^{\dagger,1}$	25841	0.057	50.4
4	0	$y^{\dagger,2}$	25841	0.047	53.6

Table 1. Some numerical data.

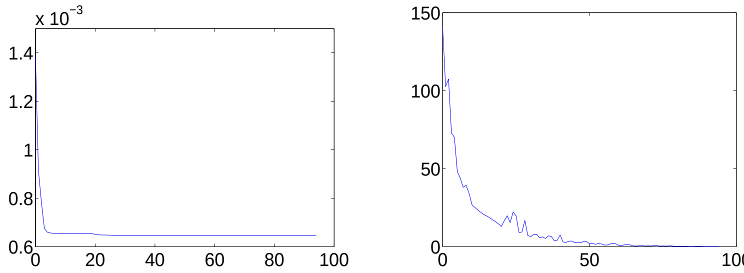

FIGURE 3. Convergence histories for the objective function $j(\psi_n)$ (left) and the angle θ_n expressed in degrees (right) for case 2.

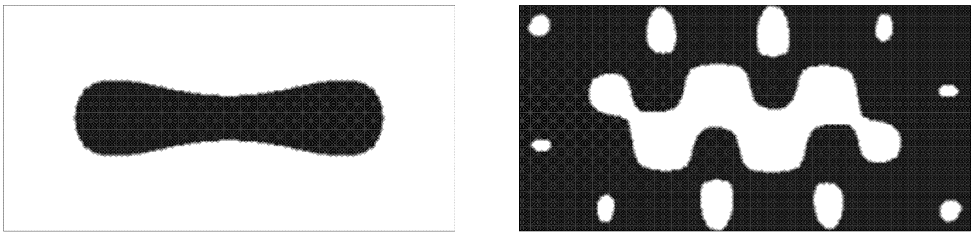

FIGURE 4. Obtained domains for $\nu = 2 \times 10^{-3}$, $y^\dagger = y^{\dagger,1}$ (case 3, left) and $\nu = 0$, $y^\dagger = y^{\dagger,2}$ (case 4, right).

Appendix A. Variation of integrals defined over level sets

The goal of this section is to provide a formula expressing the variation of an integral defined over a level set when the level set function is perturbed. The general result is stated in Theorem A.3, which expresses this variation as an integral. Then, we prove that, under additional assumptions, the integrand is continuous (Proposition A.4), which leads to a differentiability result (Corollary A.5).

We begin by recalling a formula of change of variables, which is a consequence of the coarea formula (see [13] Section 3.4.3).

Theorem A.1. *Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be Lipschitz, $n \geq 1$. Then for each $\mathcal{L}^n$-summable function $f : \mathbb{R}^n \to \mathbb{R}$ there holds*

$$\int_{\mathbb{R}^n} f |\nabla \psi| dx = \int_{\mathbb{R}} \left[\int_{[\psi = s]} f d\mathcal{H}^{n-1} \right] ds.$$

Lemma A.2. *Let $t_1 \leq t_2$ and $\psi : \mathbb{R}^n \to \mathbb{R}$ be Lipschitz with*

$$|\nabla \psi| \geq m > 0 \ a.e. \ in \ A = [t_1 \leq \psi \leq t_2].$$

Then for each function $f : \mathbb{R}^n \to \mathbb{R}$ $\mathcal{L}^n$-summable on A there holds

$$\int_A f dx = \int_{t_1}^{t_2} \left[\int_{[\psi = s]} \frac{f}{|\nabla \psi|} d\mathcal{H}^{n-1} \right] ds.$$

Proof. We define the auxiliary function

$$g = \begin{cases} |\nabla \psi| & \text{in } A, \\ 1 & \text{elsewhere.} \end{cases}$$

We have obviously

$$\int_A f dx = \int_{\mathbb{R}^n} \chi_A \frac{f}{g} |\nabla \psi| dx,$$

where χ_A is the characteristic function of A. The function $\chi_A \frac{f}{g}$ is $\mathcal{L}^n$-summable on $\mathbb{R}^n$. Hence Theorem A.1 entails

$$\int_A f dx = \int_{\mathbb{R}} \left[\int_{[\psi = s]} \chi_A \frac{f}{g} d\mathcal{H}^{n-1} \right] ds = \int_{t_1}^{t_2} \left[\int_{[\psi = s]} \frac{f}{|\nabla \psi|} d\mathcal{H}^{n-1} \right] ds.$$

$\square$

Theorem A.3. *Let $\psi, \varphi : \mathbb{R}^n \to \mathbb{R}$ be Lipschitz and bounded with $|\nabla\varphi|$ bounded. We assume that there exist positive numbers m and η such that*

$$|\nabla\psi| \geq m \qquad \mathcal{L}^n - a.e. \ in \ [|\psi| \leq \eta]. \tag{A.1}$$

Then for each function $f : \mathbb{R}^n \to \mathbb{R}$ $\mathcal{L}^n$-summable the function

$$F : t \mapsto \int_{[\psi < t\varphi]} f dx \tag{A.2}$$

admits the variation

$$F(t) - F(0) = \int_0^t \left[\int_{[\psi = s\varphi]} \frac{f\varphi}{|\nabla\psi - s\nabla\varphi|} d\mathcal{H}^{n-1} \right] ds$$

for all t in a neighborhood of 0.

Proof. Possibly making the splitting $f = f^+ - f^-$, where f^+ and f^- are the positive and negative parts of f, respectively, we assume that $f \geq 0$. We also assume that $t > 0$, the case $t < 0$ being treated in a symmetric way and the case $t = 0$ being trivial. We split the proof in three steps.

(1) A straightforward calculation provides

$$\begin{aligned} F(t) - F(0) &= \int_{[0 \leq \psi < t\varphi]} f dx - \int_{[t\varphi \leq \psi < 0]} f dx \\ &= \int_{[\varphi > 0 \ \wedge \ 0 \leq \frac{\psi}{\varphi} < t]} f dx - \int_{[\varphi < 0 \ \wedge \ 0 < \frac{\psi}{\varphi} \leq t]} f dx. \end{aligned}$$

We fix an arbitrary $\varepsilon > 0$ and define the function

$$\theta_\varepsilon = \begin{cases} \frac{\psi}{\varphi} & \text{if} \quad |\varphi| > \varepsilon, \\ \frac{\psi}{\varepsilon} & \text{if} \quad |\varphi| \leq \varepsilon. \end{cases}$$

Then we write the decomposition

$$F(t) - F(0) = F_1(\varepsilon, t) + F_2(\varepsilon, t) - F_3(\varepsilon, t) - F_4(\varepsilon, t),$$

with

$$F_1(\varepsilon, t) = \int_{[0 \leq \theta_\varepsilon < t]} f\chi_{[\varphi > \varepsilon]} dx \quad , \quad F_2(\varepsilon, t) = \int_{[0 < \varphi \leq \varepsilon \ \wedge \ 0 \leq \frac{\psi}{\varphi} < t]} f dx, \tag{A.3}$$

$$F_3(\varepsilon, t) = \int_{[0 < \theta_\varepsilon \leq t]} f\chi_{[\varphi < -\varepsilon]} dx \quad , \quad F_4(\varepsilon, t) = \int_{[-\varepsilon \leq \varphi < 0 \ \wedge \ 0 < \frac{\psi}{\varphi} \leq t]} f dx. \tag{A.4}$$

(2) It is a simple exercise to check that the function θ_ε is Lipschitz. Moreover, there exists a positive constant λ independent of ε such that $|\theta_\varepsilon| \geq \lambda|\psi|$ provided that ε stays in a bounded set, say $\varepsilon \in (0, \varepsilon_0)$. Hence, $|\nabla\psi| \geq m$ a.e. in $[|\theta_\varepsilon| \leq \lambda\eta]$. Next, we have

$$\nabla\theta_\varepsilon = \begin{cases} \dfrac{1}{\varphi}(\nabla\psi - \theta_\varepsilon\nabla\varphi) & \text{a.e. in} \quad [|\varphi| > \varepsilon], \\ \dfrac{1}{\varepsilon}\nabla\psi & \text{a.e. in} \quad [|\varphi| < \varepsilon]. \end{cases}$$

In addition, we know that $\mathcal{L}^n([|\varphi| = \varepsilon]) = 0$ for almost all $\varepsilon \in (0, \varepsilon_0)$. Indeed,

$$\int_0^{\varepsilon_0} \mathcal{L}^n([|\varphi| = \varepsilon]) d\varepsilon = \int_0^{\varepsilon_0} \left[\int_{\mathbb{R}^n} \chi_{[|\varphi| = \varepsilon]} dx \right] d\varepsilon = \int_{\mathbb{R}^n} \left[\int_0^{\varepsilon_0} \chi_{\{|\varphi(x)|\}} d\varepsilon \right] dx = 0.$$

We assume that ε is appropriately chosen. We deduce that $|\nabla\theta_\varepsilon| \geq \mu > 0$ a.e. in $[|\theta_\varepsilon| \leq t]$, provided that t is small enough, say $t \leq t_0$, with t_0 independent of ε. By Lemma A.2, there holds

$$F_1(\varepsilon, t) - F_3(\varepsilon, t) = \int_0^t \left[\int_{[\theta_\varepsilon = s]} \frac{f\chi_{[\varphi > \varepsilon]}}{|\nabla\theta_\varepsilon|} d\mathcal{H}^{n-1} \right] ds - \int_0^t \left[\int_{[\theta_\varepsilon = s]} \frac{f\chi_{[\varphi < -\varepsilon]}}{|\nabla\theta_\varepsilon|} d\mathcal{H}^{n-1} \right] ds.$$

In both integrands there holds $\theta_\varepsilon = \psi/\varphi = s$ and

$$\nabla\theta_\varepsilon = \frac{\nabla\psi - s\nabla\varphi}{\varphi}.$$

It comes

$$F_1(\varepsilon,t)-F_3(\varepsilon,t)=\int_0^t\left[\int_{[\psi=s\varphi]}\frac{f\varphi\chi_{[\varphi>\varepsilon]}}{|\nabla\psi-s\nabla\varphi|}d\mathcal{H}^{n-1}\right]ds+\int_0^t\left[\int_{[\psi=s\varphi]}\frac{f\varphi\chi_{[\varphi<-\varepsilon]}}{|\nabla\psi-s\nabla\varphi|}d\mathcal{H}^{n-1}\right]ds. \quad (A.5)$$

By using the expressions (A.3) and (A.4), it turns out that each term in (A.5) is bounded in absolute value by $\int_{\mathbb{R}^n} f dx$. By the monotone convergence theorem, we obtain

$$\begin{aligned}\lim_{\varepsilon\to 0} F_1(\varepsilon,t)-F_3(\varepsilon,t) &= \int_0^t\left[\int_{[\psi=s\varphi]}\frac{f\varphi\chi_{[\varphi>0]}}{|\nabla\psi-s\nabla\varphi|}d\mathcal{H}^{n-1}\right]ds+\int_0^t\left[\int_{[\psi=s\varphi]}\frac{f\varphi\chi_{[\varphi<0]}}{|\nabla\psi-s\nabla\varphi|}d\mathcal{H}^{n-1}\right]ds\\ &= \int_0^t\left[\int_{[\psi=s\varphi]}\frac{f\varphi}{|\nabla\psi-s\nabla\varphi|}d\mathcal{H}^{n-1}\right]ds.\end{aligned}$$

(3) Besides, we have

$$|F_2(\varepsilon,t)-F_4(\varepsilon,t)|\le\int_{[|\psi|\le t\varepsilon]}|f|dx=\int_{\mathbb{R}^n}\chi_{[|\psi|\le t\varepsilon]}|f|dx.$$

Yet, since $\mathcal{L}^n([\psi=0])=0$ (which can be deduced from Lemma A.2 but is quite obvious in view of (A.1)), there holds $\lim_{\varepsilon\to 0}\chi_{[|\psi|\le t\varepsilon]}=0$ a.e. in $\mathbb{R}^n$. Hence the dominated convergence theorem yields

$$\lim_{\varepsilon\to 0}F_2(\varepsilon,t)-F_4(\varepsilon,t)=0,$$

which completes the proof.

□

Proposition A.4. *Let D be an open and bounded subset of $\mathbb{R}^2$ and $\psi,\varphi:\overline{D}\to\mathbb{R}$ be two functions of class $\mathcal{C}^1$. We assume that $[\psi=0]\subset\subset D$ and $\nabla\psi$ does not vanish on $[\psi=0]$. Let $f:D\to\mathbb{R}$ be continuous in the vicinity of $[\psi=0]$. Then the function*

$$G: t\mapsto\int_{[\psi=t\varphi]}fd\mathcal{H}^1$$

is continuous in the vicinity of 0.

Proof. By the implicit function theorem together with the compactness of the set $[\psi=0]$, there exist a family of open sets $(V_i)_{i=1,\dots,N}$, a positive number $\bar{t}$ and $\mathcal{C}^1$ functions $\Phi_i:\mathbb{R}\to\mathbb{R}$ such that $\cup_{i=1}^N V_i\supset\bar{A}$, where A is an open tubular neighborhood of $[\psi=0]$, and

$$\forall i=1,...,N,\ \forall (x,t)\in V_i\times(-\bar{t},\bar{t}),\ \psi(x)=t\varphi(x)\Leftrightarrow x_2=\Phi_i(x_1,t)$$

Here, (x_1,x_2) stands for the representation of x in an appropriate local Cartesian coordinate system. Also, by compactness of $\bar{D}\setminus A$ and continuity of ψ and φ, for t sufficiently small, $\psi(x)=t\varphi\Rightarrow x\in A$. We adjust $\bar{t}$ in this respect. Now, let $(\eta_i)_{i=0,\dots,N}$ be a partition of unity of $\bar{A}$ subordinate to the cover $(V_i)_{i=0,\dots,N}$. We have for all $t\in(-\bar{t},\bar{t})$

$$G(t)=\sum_{i=1}^N\int_{[\psi=t\varphi]}\eta_i f d\mathcal{H}^1=\sum_{i=1}^N\int_{\mathbb{R}}(\eta_i f)(x_1,\Phi_i(t,x_1))\sqrt{1+\frac{\partial\Phi_i}{\partial x_1}(t,x_1)^2}dx_1.$$

We deduce the continuity of G by standard arguments. □

Corollary A.5. *Let ψ, φ, f be functions satisfying the assumptions of Proposition A.4, with f $\mathcal{L}^2$-summable in D. Then the function F defined by* (A.2) *is differentiable in a neighborhood of* 0. *In particular,*

$$F'(0)=\int_{[\psi=0]}\frac{f\varphi}{|\nabla\psi|}ds.$$

Proof. By Theorem A.3, we have for all t in a neighborhood of 0

$$F(t)-F(0)=\int_0^t G(s)ds,$$

with

$$G(s)=\int_{[\psi=s\varphi]}\frac{f\varphi}{|\nabla\psi-s\nabla\varphi|}d\mathcal{H}^1.$$

With the help of Proposition A.4, we obtain that G is continuous in a neighborhood of 0, which straightforwardly yields the result. □

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