# REGULARIZED PERIMETER FOR TOPOLOGY OPTIMIZATION[*]

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**Abstract.** The perimeter functional is known to oppose serious difficulties when it has to be handled within a topology optimization procedure. In this paper, a regularized perimeter functional $\text{Per}_\varepsilon$, defined for two- and three-dimensional domains, is introduced. On one hand, the convergence of $\text{Per}_\varepsilon$ to the exact perimeter when $\varepsilon$ tends to zero is proved. On the other hand, the topological differentiability of $\text{Per}_\varepsilon$ for $\varepsilon > 0$ is analyzed. These features lead to the design of a topology optimization algorithm suitable for perimeter-dependent objective functionals. Several numerical results illustrate the method.

**1. Introduction.** Topology optimization problems are known to be generally ill-posed, in the sense that they possess no global minimizers. Typically, this property stems from the fact that the minimizing sequences have more and more complex topologies, without ever converging to a domain in any appropriate way [2, 16]. Therefore, relaxation methods are often used [1, 9, 12], but the binary nature of the problem is then lost. A totally different approach is to impose geometrical constraints that limit the complexity of the obtained topologies. In this framework, a classical technique is to incorporate in the cost function a penalization by the perimeter. In many important cases, the resulting problem can be proved to be well-posed [4, 11, 16]. The control of the perimeter of domains with variable topology appears also in image processing, for instance, when considering the Mumford–Shah functional [20].

From a practical point of view, the perimeter functional can be relatively easily handled by boundary variation methods, as its shape derivative is properly defined as the mean curvature of the boundary [3, 24]. However, serious difficulties are encountered as soon as one wants to perform topology changes. To illustrate this, let us consider the creation of a small hole $\omega_\rho = B(z, \rho)$ inside a domain $\Omega \subset \mathbb{R}^N$ seen as the current design domain in an iterative process. Then the variation of the perimeter is given by $\text{Per}(\Omega \setminus \overline{\omega_\rho}) - \text{Per}(\Omega) = \text{Per}(\omega_\rho) = \rho^{N-1} \text{Per}(\omega_1)$. In contrast, the variation of the volume is $|\Omega \setminus \overline{\omega_\rho}| - |\Omega| = |\omega_\rho| = \rho^N |\omega_1|$. In fact, the traditional shape functionals, such as the compliance, also admit a first variation proportional to $\rho^N$, at least when Neumann boundary conditions are prescribed on $\partial\omega_\rho$ [5, 13, 23]. This difference of order of convergence prevents a successful numerical treatment of the perimeter in conjunction with other shape functionals by methods based on small topology perturbations. To circumvent this difficulty, a two-step algorithm is used in [17]: a "topological" step which does not take into account the perimeter, and then a "classical" step based on smooth boundary variation methods. The basic ingredients in each step are the notions of topological and shape derivatives, respectively.

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More sophisticated approaches, also based on alternating steps, have been proposed in [14, 15].

In this paper, we present a natural way to include the perimeter within a topology optimization procedure. The proposed approach is based on a regularization method: the perimeter $\mathrm{Per}(\Omega)$ is approximated by a functional $\mathrm{Per}_\varepsilon(\Omega)$ well-suited for topology optimization, and then $\varepsilon$ is driven to zero, for which the exact perimeter is retrieved. Let us give a little more detail. Let $\Omega$ be an open and bounded subset of $\mathbb{R}^N$, $N \in \{2,3\}$, with $\mathcal{C}^2$ boundary $\partial\Omega$. We denote by $u$ the characteristic function of $\Omega$, i.e., $u(x) = 1$ if $x \in \Omega$, $u(x) = 0$ if $x \in \mathbb{R}^N \setminus \Omega$. For a fixed $m \in \mathbb{N}^\star$ and any $\varepsilon > 0$ we consider the (weak) solution $u_\varepsilon \in H^m(\mathbb{R}^N)$ of

(1.1)
$$\varepsilon^{2m}(-\Delta)^m u_\varepsilon + u_\varepsilon = u.$$

Then we define the quantity

$$E_\varepsilon(\Omega) := \|u - u_\varepsilon\|_{L^2(\mathbb{R}^N)}^2 = \int_{\mathbb{R}^N} (u - u_\varepsilon)^2 dx.$$

We shall see that the asymptotic behavior of $E_\varepsilon(\Omega)$ when $\varepsilon$ goes to zero is directly related to the perimeter of $\Omega$. Before giving a precise statement, let us specify some notation. We denote by $\langle .,. \rangle$ the canonical scalar product of $\mathbb{R}^N$ and by $|.|$ the associated norm. For complex vectors, the same notation is kept for the Hermitian scalar product of $\mathbb{C}^N$ and its norm, while complex conjugacy is denoted by a bar. The surface measure on $\partial\Omega$ is denoted by $\sigma$. Therefore, the perimeter of $\Omega$ can be defined as

$$\mathrm{Per}(\Omega) = \sigma(\partial\Omega) = \int_{\partial\Omega} d\sigma.$$

The outward unit normal to $\partial\Omega$ at point $x$ is denoted by $n(x)$. We shall prove the following result.

THEOREM 1.1. *The following asymptotic expansion holds when $\varepsilon$ goes to zero:*

$$E_\varepsilon(\Omega) = \varepsilon \kappa_m \mathrm{Per}(\Omega) + O(\varepsilon^{\frac{N+4}{N+2}}),$$

*where the constant $\kappa_m$ is defined by*

$$\kappa_m = \frac{1}{\pi} \int_0^\infty \frac{t^{4m-2}}{(1 + t^{2m})^2} dt.$$

*The first values of $\kappa_m$ are $\kappa_1 = 1/4$ and $\kappa_2 = 3/2^{7/2}$.*

Therefore, we call regularized perimeter the quantity

(1.2)
$$\mathrm{Per}_\varepsilon(\Omega) = \frac{1}{\kappa_m \varepsilon} E_\varepsilon(\Omega),$$

which, by the consequences of Theorem 1.1, satisfies

$$\mathrm{Per}_\varepsilon(\Omega) = \mathrm{Per}(\Omega) + O(\varepsilon^{\frac{2}{N+2}}).$$

Theorem 1.1 is proved in section 2. In section 3, the result is extended to a boundary value problem, where (1.1) is complemented by a Neumann boundary condition on the border of a bounded domain $D$ containing $\Omega$. In section 4, the sensitivity of

the functional $\mathrm{Per}_\varepsilon$ to topological perturbations is analyzed. In particular, we establish that the first variation of the regularized functional is proportional to the volume of the perturbation. Then, in section 5, we show how these results lead to a topology optimization algorithm dedicated to perimeter-dependent objective functionals. Some numerical experiments are reported in section 6. Concluding remarks are given in section 7.

**2. Asymptotic expansion of the regularized functional.** This section is devoted to the proof of Theorem 1.1. Our approach relies on the Fourier transform, for which we adopt the definition

$$\forall f \in L^1(\mathbb{R}^N), \qquad \hat{f}(\xi) = (2\pi)^{-N/2} \int_{\mathbb{R}^N} e^{-i\langle x, \xi \rangle} f(x) dx.$$

For a detailed exposition of the Fourier transform properties, we refer the reader to, e.g., [18].

**2.1. Reformulation in the frequency domain.** Passing to the Fourier transform in (1.1) yields

$$\varepsilon^{2m} |\xi|^{2m} \hat{u}_\varepsilon(\xi) + \hat{u}_\varepsilon(\xi) = \hat{u}(\xi),$$

from which we derive

$$\hat{u}_\varepsilon(\xi) = \frac{\hat{u}(\xi)}{1 + (\varepsilon|\xi|)^{2m}}.$$

Next, by Parseval's equality, we obtain

$$E_\varepsilon(\Omega) = \|\hat{u} - \hat{u}_\varepsilon\|_{L^2(\mathbb{R}^N)}^2 = \int_{\mathbb{R}^N} \left( \frac{(\varepsilon|\xi|)^{2m}}{1 + (\varepsilon|\xi|)^{2m}} \right)^2 |\hat{u}(\xi)|^2 d\xi.$$

The change of variable $\zeta = \varepsilon\xi$ results in

$$(2.1) \qquad E_\varepsilon(\Omega) = \varepsilon^{-N} \int_{\mathbb{R}^N} \frac{|\zeta|^{4m}}{(1 + |\zeta|^{2m})^2} |\hat{u}(\varepsilon^{-1}\zeta)|^2 d\zeta.$$

It will turn out to be useful—and of independent interest—to study a generalized version of (2.1). To this aim, for all $k \in \mathbb{N}$, we introduce the linear space

$$\mathcal{V}_k = \left\{ \Phi \in \mathcal{C}^\infty(\mathbb{R}), [t \mapsto t^{k-2}(1+t^2)^2 \Phi^{(k)}(t)] \in L^\infty(\mathbb{R}) \right\},$$

endowed with the seminorm

$$\|\Phi\|_{\mathcal{V}_k} = \left\| t \mapsto t^{k-2}(1+t^2)^2 \Phi^{(k)}(t) \right\|_{L^\infty(\mathbb{R})} = \inf \left\{ a \in \mathbb{R}, |\Phi^{(k)}(t)| \leq a \frac{|t|^{2-k}}{(1+t^2)^2} \ \forall t \in \mathbb{R} \right\}.$$

Then, for all $\Phi \in \mathcal{V}_0$, we set

$$(2.2) \qquad T_\varepsilon(\Phi) = \varepsilon^{-N} \int_{\mathbb{R}^N} \Phi(|\zeta|) |\zeta|^2 |\hat{u}(\varepsilon^{-1}\zeta)|^2 d\zeta.$$

Since $\hat{u} \in L^2(\mathbb{R}^N)$, the above expression makes sense for all $\Phi \in \mathcal{V}_0$, and furthermore $T_\varepsilon$ belongs to $\mathcal{V}_0'$, the space of continuous linear functionals on $\mathcal{V}_0$. We also define the linear functional $\tilde{T}_\varepsilon \in \mathcal{V}_0'$ by

$$\tilde{T}_\varepsilon(\Phi) = \frac{\varepsilon}{\pi} \mathrm{Per}(\Omega) \int_0^\infty \Phi(t) dt,$$

and define the linear space $\mathcal{V}$ by

$$\mathcal{V} = \bigcap_{k=0}^{N+1} \mathcal{V}_k$$

endowed with the norm

$$\|\Phi\|_{\mathcal{V}} = \max\left\{\|\Phi\|_{\mathcal{V}_k}, k = 0, \ldots, N+1\right\}.$$

We shall prove the following result.

THEOREM 2.1. *There exists $c > 0$ such that, for all $\Phi \in \mathcal{V}$ and all $\varepsilon$ sufficiently small,*

$$|T_\varepsilon(\Phi) - \tilde{T}_\varepsilon(\Phi)| \leq c\varepsilon^{\frac{N+4}{N+2}}\|\Phi\|_{\mathcal{V}}.$$

Then Theorem 1.1 follows at once from Theorem 2.1 by choosing

$$\Phi(t) = \frac{t^{4m-2}}{(1 + t^{2m})^2}.$$

We need only to check that this function belongs to $\mathcal{V}$. To do so we set $G_k(t) = t^{2-k}/(1+t^2)^2$. We remark that $\Phi^{(k)}/G_k$ is a rational function of degree 0, and hence it will be bounded as soon as it has no pole on the real line. Immediate calculations provide

$$\frac{\Phi(t)}{G_0(t)} = \left[t^{2m-2}\frac{1+t^2}{1+t^{2m}}\right]^2,$$

$$\frac{\Phi'(t)}{G_1(t)} = (4m-2)\left[t^{2m-2}\frac{1+t^2}{1+t^{2m}}\right]^2 - 4mt^{6m-4}\frac{(1+t^2)^2}{(1+t^{2m})^3},$$

$$\forall k \geq 2, \quad \frac{\Phi^{(k)}(t)}{G_k(t)} = \Phi^{(k)}(t)\, t^{k-2}(1+t^2)^2.$$

Obviously the above rational functions have no real poles for any $m \geq 1$.

The rest of this section is devoted to the proof of Theorem 2.1. Throughout, the letter $c$ will be used to denote any positive constant independent of $\varepsilon$ and $\Phi$. The proof is divided into three parts.

**2.2. Derivation of the leading term.** First, we assume that $\Phi \in \mathcal{C}_0^\infty(\mathbb{R})$, the set of functions of class $\mathcal{C}^\infty$ on $\mathbb{R}$ with compact support. By definition we have

$$|\xi|^2\hat{u}(\xi) = (2\pi)^{-N/2}|\xi|^2\int_\Omega e^{-i\langle x,\xi\rangle}\,dx = (2\pi)^{-N/2}\int_\Omega \mathrm{div}_x\left(ie^{-i\langle x,\xi\rangle}\xi\right)dx,$$

which, by the divergence formula and setting $e_\xi = \xi/|\xi|$, yields

(2.3) $$|\xi|\hat{u}(\xi) = (2\pi)^{-N/2}i\int_{\partial\Omega} e^{-i\langle x,\xi\rangle}\langle e_\xi, n(x)\rangle d\sigma(x).$$

On writing $|\hat{u}(\xi)|^2 = \hat{u}(\xi)\overline{\hat{u}(\xi)}$ we obtain from (2.3)

$$|\xi|^2|\hat{u}(\xi)|^2 = (2\pi)^{-N}\int_{\partial\Omega\times\partial\Omega} e^{-i\langle x-y,\xi\rangle}\langle e_\xi, n(x)\rangle\langle e_\xi, n(y)\rangle d\sigma(x)d\sigma(y).$$

Plugging this expression into (2.2) entails

$$T_\varepsilon(\Phi) = (2\pi)^{-N} \varepsilon^{2-N} \int_{\mathbb{R}^N} \Phi(|\zeta|) \left[ \int_{\partial\Omega \times \partial\Omega} e^{i\varepsilon^{-1}\langle y-x,\zeta\rangle} \langle e_\zeta, n(x)\rangle \langle e_\zeta, n(y)\rangle d\sigma(x) d\sigma(y) \right] d\zeta.$$

By Fubini's theorem, this can be reordered as

$$T_\varepsilon(\Phi) = (2\pi)^{-N} \varepsilon^{2-N} \int_{\partial\Omega} \left\langle \int_{\partial\Omega} \left( \int_{\mathbb{R}^N} e^{i\varepsilon^{-1}\langle y-x,\zeta\rangle} \Phi(|\zeta|) e_\zeta \otimes e_\zeta d\zeta \right) n(y) d\sigma(y), n(x) \right\rangle d\sigma(x).$$

Setting

$$(2.4) \qquad \varphi(z) = (2\pi)^{-N/2} \int_{\mathbb{R}^N} e^{i\langle z,\zeta\rangle} \Phi(|\zeta|) e_\zeta \otimes e_\zeta d\zeta,$$

$$(2.5) \qquad F_\varepsilon(x) = \int_{\partial\Omega} \varphi(\varepsilon^{-1}(y-x)) n(y) d\sigma(y),$$

we arrive at

$$(2.6) \qquad T_\varepsilon(\Phi) = (2\pi)^{-N/2} \varepsilon^{2-N} \int_{\partial\Omega} \langle F_\varepsilon(x), n(x)\rangle d\sigma(x).$$

We shall now examine the asymptotic behavior of $F_\varepsilon(x)$ for a given $x \in \partial\Omega$. Let $\rho > 0$ be such that the set $\partial\Omega \cap B(x, 2\rho)$ can be represented as the graph of a $\mathcal{C}^2$ function on an appropriate local Cartesian coordinate system. Note that, by compactness of $\partial\Omega$, $\rho$ may be chosen independent of $x$. Let $\eta : \mathbb{R}^N \to \mathbb{R}$ be a smooth ($\mathcal{C}^\infty$) function such that $\eta(z) = 1$ if $|z| \leq \rho$, $0 \leq \eta(z) \leq 1$ if $\rho \leq |z| \leq 2\rho$, and $\eta(z) = 0$ if $|z| \geq 2\rho$. We introduce a parameter $\beta \in ]0, 1[$, which will be fixed later, and split (2.5) as

$$(2.7) \qquad F_\varepsilon(x) = F_\varepsilon^0(x) + F_\varepsilon^1(x),$$

with

$$(2.8) \qquad F_\varepsilon^0(x) = \int_{\partial\Omega} \eta(\varepsilon^{-\beta}(y-x)) \varphi(\varepsilon^{-1}(y-x)) n(y) d\sigma(y),$$

$$(2.9) \qquad F_\varepsilon^1(x) = \int_{\partial\Omega} [1 - \eta(\varepsilon^{-\beta}(y-x))] \varphi(\varepsilon^{-1}(y-x)) n(y) d\sigma(y).$$

In the ball $B(x, 2\rho)$ we parametrize $\partial\Omega$ by

$$(2.10) \qquad t \in \mathcal{O} \mapsto y(t) = x + R(t, \psi(t)) \in \partial\Omega,$$

where $\mathcal{O}$ is an open set of $\mathbb{R}^{N-1}$ containing the origin, $R$ is a rotation, and $\psi : \mathcal{O} \to \mathbb{R}$ is a function of class $\mathcal{C}^2$ satisfying

$$(2.11) \qquad \psi(0) = 0, \qquad \nabla\psi(0) = 0.$$

For notational simplicity, we write vectors of $\mathbb{R}^N$ indifferently row-wise or column-wise. We subsequently assume that $\varepsilon < 1$. Then $\eta(\varepsilon^{-\beta}(y-x)) \neq 0$ implies $y \in B(x, 2\rho)$, and we can write

$$F_\varepsilon^0(x) = \int_{\mathcal{O}} \eta(\varepsilon^{-\beta}(y(t)-x)) \varphi(\varepsilon^{-1}(y(t)-x)) R(-\nabla\psi(t), 1) dt.$$

Setting

$$\eta_\varepsilon(t) = \begin{cases} \eta(\varepsilon^{-\beta}(y(t) - x)) & \text{if } t \in \mathcal{O}, \\ 0 & \text{otherwise,} \end{cases}$$

we obtain

$$F_\varepsilon^0(x) = \int_{\mathbb{R}^{N-1}} \eta_\varepsilon(t)\varphi(R(\varepsilon^{-1}t, \varepsilon^{-1}\psi(t)))R(-\nabla\psi(t), 1)dt.$$

By the definition (2.4), we observe that

(2.12) $$\varphi(Rz) = R\varphi(z)R^\star,$$

with $R^\star$ the adjoint operator of $R$, i.e., the rotation of opposite angle. This entails

$$F_\varepsilon^0(x) = R \int_{\mathbb{R}^{N-1}} \eta_\varepsilon(t)\varphi(\varepsilon^{-1}t, \varepsilon^{-1}\psi(t))(-\nabla\psi(t), 1)dt.$$

Then, by the change of variable $t = \varepsilon s$, we arrive at

(2.13) $$F_\varepsilon^0(x) = \varepsilon^{N-1}R \int_{\mathbb{R}^{N-1}} \eta_\varepsilon(\varepsilon s)\varphi(s, \varepsilon^{-1}\psi(\varepsilon s))(-\nabla\psi(\varepsilon s), 1)ds.$$

Let $e_N = (0, \ldots, 0, 1)$ be the last vector of the canonical basis of $\mathbb{R}^N$. We split (2.13) as

(2.14) $$F_\varepsilon^0(x) = A_\varepsilon(x) + B_\varepsilon(x) + C_\varepsilon(x) + D_\varepsilon(x),$$

with

(2.15) $$A_\varepsilon(x) = \varepsilon^{N-1}R \int_{\mathbb{R}^{N-1}} \varphi(s, 0)e_N ds,$$

(2.16) $$B_\varepsilon(x) = \varepsilon^{N-1}R \int_{\mathbb{R}^{N-1}} [\eta_\varepsilon(\varepsilon s) - 1]\varphi(s, 0)e_N ds,$$

(2.17) $$C_\varepsilon(x) = \varepsilon^{N-1}R \int_{\mathbb{R}^{N-1}} \eta_\varepsilon(\varepsilon s)\left[\varphi(s, \varepsilon^{-1}\psi(\varepsilon s)) - \varphi(s, 0)\right]e_N ds,$$

(2.18) $$D_\varepsilon(x) = \varepsilon^{N-1}R \int_{\mathbb{R}^{N-1}} \eta_\varepsilon(\varepsilon s)\varphi(s, \varepsilon^{-1}\psi(\varepsilon s))(-\nabla\psi(\varepsilon s), 0)ds.$$

We first focus on the expected leading term $A_\varepsilon(x)$. We define for every $\zeta \in \mathbb{R}^N$ and $\zeta' \in \mathbb{R}^{N-1}$

(2.19) $$h(\zeta) = \Phi(|\zeta|)e_\zeta \otimes e_\zeta, \qquad H(\zeta') = \int_{\mathbb{R}} h(\zeta', \zeta_N)d\zeta_N.$$

Therefore the definition (2.4) is equivalent to $\varphi(z) = \overline{\hat{h}(z)}$. In addition, the $(N-1)$-dimensional Fourier transform of $H$ is given, for all $s \in \mathbb{R}^{N-1}$, by

$$\hat{H}(s) = (2\pi)^{-\frac{N-1}{2}} \int_{\mathbb{R}^{N-1}} e^{-i\langle s, \zeta' \rangle} \int_{\mathbb{R}} h(\zeta', \zeta_N)d\zeta_N d\zeta' = (2\pi)^{\frac{1}{2}}\hat{h}(s, 0),$$

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and thus

$$\varphi(s,0) = (2\pi)^{-\frac{1}{2}}\overline{\hat{H}(s)}.$$

It follows that

$$A_\varepsilon(x) = (2\pi)^{-\frac{1}{2}}\varepsilon^{N-1}R\left[\int_{\mathbb{R}^{N-1}} \overline{\hat{H}(s)}ds\right]e_N.$$

Next, the Fourier inversion formula yields

$$A_\varepsilon(x) = (2\pi)^{\frac{N}{2}-1}\varepsilon^{N-1}R\overline{H(0)}e_N.$$

From

$$H(0) = \int_{\mathbb{R}} \Phi(|\zeta_N|)e_N \otimes e_N d\zeta_N = 2\left(\int_0^\infty \Phi(t)dt\right)e_N \otimes e_N,$$

we arrive at

$$A_\varepsilon(x) = (2\pi)^{\frac{N}{2}-1}\varepsilon^{N-1}2\left(\int_0^\infty \Phi(t)dt\right)Re_N = (2\pi)^{\frac{N}{2}-1}\varepsilon^{N-1}2\left(\int_0^\infty \Phi(t)dt\right)n(x).$$

The contribution of $A_\varepsilon(x)$ in the functional $E_\varepsilon(\Omega)$ is then given by

(2.20)
$$(2\pi)^{-N/2}\varepsilon^{2-N}\int_{\partial\Omega}\langle A_\varepsilon(x), n(x)\rangle d\sigma(x) = (2\pi)^{-1}\varepsilon 2\left(\int_0^\infty \Phi(t)dt\right)\int_{\partial\Omega}\langle n(x), n(x)\rangle d\sigma(x)$$

(2.21)
$$= \frac{\varepsilon}{\pi}\left(\int_0^\infty \Phi(t)dt\right)\text{Per}(\Omega) = \tilde{T}_\varepsilon(\Phi).$$

**2.3. Estimate of remainders.** From (2.6) and (2.20) we find

$$T_\varepsilon(\Phi) - \tilde{T}_\varepsilon(\Phi) = (2\pi)^{-N/2}\varepsilon^{2-N}\int_{\partial\Omega}\langle F_\varepsilon(x) - A_\varepsilon(x), n(x)\rangle d\sigma(x).$$

Then using (2.7) and (2.14), we arrive at
(2.22)
$$T_\varepsilon(\Phi) - \tilde{T}_\varepsilon(\Phi) = (2\pi)^{-N/2}\varepsilon^{2-N}\int_{\partial\Omega}\langle F_\varepsilon^1(x) + B_\varepsilon(x) + C_\varepsilon(x) + D_\varepsilon(x), n(x)\rangle d\sigma(x).$$

We shall estimate each term of the integrand in (2.22). Beforehand, we shall establish useful estimates for the function $\varphi$ defined by (2.4).

By successive integrations by parts from (2.4), we obtain for each $j \in \{1, \ldots, N\}$ and any $n \in \mathbb{N}$

(2.23)
$$\varphi(z)z_j^n = (2\pi)^{-N/2}i^n\int_{\mathbb{R}^N} e^{i\langle z,\zeta\rangle}\frac{\partial^n}{\partial\zeta_j^n}\left(\Phi(|\zeta|)e_\zeta \otimes e_\zeta\right)d\zeta.$$

Here, $z_j$ and $\zeta_j$ stand for the $j$th components of the vectors $z$ and $\zeta$, respectively. The Leibniz formula provides
(2.24)
$$\frac{\partial^n}{\partial\zeta_j^n}\left(\Phi(|\zeta|)e_\zeta \otimes e_\zeta\right) = \frac{\partial}{\partial\zeta_j^n}\left(\frac{\Phi(|\zeta|)}{|\zeta|^2}\zeta \otimes \zeta\right) = \sum_{k=0}^2\binom{n}{k}\frac{\partial^{n-k}}{\partial\zeta_j^{n-k}}\left(\frac{\Phi(|\zeta|)}{|\zeta|^2}\right)\frac{\partial^k}{\partial\zeta_j^k}(\zeta \otimes \zeta).$$

By induction, deferred to Appendix A, we prove that

$$(2.25) \qquad \frac{\partial^q}{\partial \zeta_j^q}\left(\frac{\Phi(|\zeta|)}{|\zeta|^2}\right) = \frac{1}{|\zeta|^{2q+2}}\sum_{p=0}^{q}\Phi^{(p)}(|\zeta|)P_{p,q}(|\zeta|,\zeta_j),$$

where $P_{p,q}$ is a homogeneous polynomial of two variables of degree $p+q$. This entails

$$\left|\frac{\partial^q}{\partial \zeta_j^q}\left(\frac{\Phi(|\zeta|)}{|\zeta|^2}\right)\right| \le c_q\frac{1}{|\zeta|^{2q+2}}\sum_{p=0}^{q}|\Phi^{(p)}(|\zeta|)||\zeta|^{p+q}$$

for some constants $c_q > 0$. Using now that, by definition,

$$|\Phi^{(p)}(|\zeta|)| \le \|\Phi\|_{\mathcal{V}_p}\frac{|\zeta|^{2-p}}{(1+|\zeta|^2)^2},$$

we obtain

$$\left|\frac{\partial^q}{\partial \zeta_j^q}\left(\frac{\Phi(|\zeta|)}{|\zeta|^2}\right)\right| \le (q+1)c_q\frac{\max\left(\|\Phi\|_{\mathcal{V}_p},p\le q\right)}{|\zeta|^q(1+|\zeta|^2)^2}.$$

Plugging this estimate into (2.24), we get

$$(2.26)$$
$$\left|\frac{\partial^n}{\partial \zeta_j^n}\left(\Phi(|\zeta|)e_\zeta\otimes e_\zeta\right)\right| \le \sum_{k=0}^{2}\binom{n}{k}(n-k+1)c_{n-k}\frac{\max\left(\|\Phi\|_{\mathcal{V}_p},p\le n-k\right)}{|\zeta|^{n-k}(1+|\zeta|^2)^2}|\zeta|^{2-k}$$

$$(2.27) \qquad\qquad \le c_n\frac{\max\left(\|\Phi\|_{\mathcal{V}_p},p\le n\right)}{|\zeta|^{n-2}(1+|\zeta|^2)^2}$$

for some other constant $c_n > 0$. The combination of (2.23) and (2.26) leads to

$$|\varphi(z)||z|^n \le (2\pi)^{-N/2}c_n\max\left(\|\Phi\|_{\mathcal{V}_p},p\le n\right)\int_{\mathbb{R}^N}\frac{d\zeta}{|\zeta|^{n-2}(1+|\zeta|^2)^2}.$$

The integral on the right-hand side of the above inequality is finite whenever $N-1\le n\le N+1$. We conclude that

$$(2.28) \qquad \forall n\in\{N-1,N,N+1\},\qquad |\varphi(z)| \le c\frac{\max\left(\|\Phi\|_{\mathcal{V}_p},p\le n\right)}{|z|^n} \le c\frac{\|\Phi\|_{\mathcal{V}}}{|z|^n}.$$

Next we study the partial derivative

$$\frac{\partial\varphi}{\partial z_N}(z) = (2\pi)^{-N/2}i\int_{\mathbb{R}^N}e^{i\langle z,\zeta\rangle}\zeta_N\Phi(|\zeta|)e_\zeta\otimes e_\zeta d\zeta.$$

By successive integrations by parts we find

$$(2.29) \qquad \frac{\partial\varphi}{\partial z_N}(z)z_j^n = (2\pi)^{-N/2}i^{n+1}\int_{\mathbb{R}^N}e^{i\langle z,\zeta\rangle}\frac{\partial^n}{\partial \zeta_j^n}\left(\zeta_N\Phi(|\zeta|)e_\zeta\otimes e_\zeta\right)d\zeta.$$

If $j\ne N$, we have obviously

$$\frac{\partial\varphi}{\partial z_N}(z)z_j^n = (2\pi)^{-N/2}i^{n+1}\int_{\mathbb{R}^N}e^{i\langle z,\zeta\rangle}\zeta_N\frac{\partial^n}{\partial \zeta_j^n}\left(\Phi(|\zeta|)e_\zeta\otimes e_\zeta\right)d\zeta.$$

Using (2.26) we obtain
(2.30)
$$\forall j \neq N, \qquad \left|\frac{\partial \varphi}{\partial z_N}(z)\right| |z_j|^n \leq (2\pi)^{-N/2} c_n \max\left(\|\Phi\|_{\mathcal{V}_p}, p \leq n\right) \int_{\mathbb{R}^N} \frac{d\zeta}{|\zeta|^{n-3}(1+|\zeta|^2)^2}.$$

For $j = N$ the Leibniz formula provides

$$\frac{\partial^n}{\partial \zeta_N^n}\left(\zeta_N \Phi(|\zeta|) e_\zeta \otimes e_\zeta\right) = \zeta_N \frac{\partial^n}{\partial \zeta_N^n}\left(\Phi(|\zeta|) e_\zeta \otimes e_\zeta\right) + n \frac{\partial^{n-1}}{\partial \zeta_N^{n-1}}\left(\Phi(|\zeta|) e_\zeta \otimes e_\zeta\right).$$

Then (2.26) yields

$$\left|\frac{\partial^n}{\partial \zeta_N^n}\left(\zeta_N \Phi(|\zeta|) e_\zeta \otimes e_\zeta\right)\right| \leq (2\pi)^{-N/2}(c_n + nc_{n-1})\frac{\max\left(\|\Phi\|_{\mathcal{V}_p}, p \leq n\right)}{|\zeta|^{n-3}(1+|\zeta|^2)^2},$$

which, in view of (2.29), implies
(2.31)
$$\left|\frac{\partial \varphi}{\partial z_N}(z)\right| |z_N|^n \leq (2\pi)^{-N/2}(c_n + nc_{n-1})\max\left(\|\Phi\|_{\mathcal{V}_p}, p \leq n\right)\int_{\mathbb{R}^N} \frac{d\zeta}{|\zeta|^{n-3}(1+|\zeta|^2)^2}.$$

The two integrals on the right-hand sides of (2.30) and (2.31) are finite whenever $N \leq n \leq N+2$. We conclude that

(2.32) $$\forall n \in \{N, N+1\}, \qquad \left|\frac{\partial \varphi}{\partial z_N}(z)\right| \leq c\frac{\|\Phi\|_{\mathcal{V}}}{|z|^n}.$$

We are now in position to estimate all the remainders.
1. When $\varepsilon^{-\beta}(y-x)$ belongs to the support of $1 - \eta$, we have $\varepsilon^{-\beta}|y-x| \geq \rho$, and hence, in view of (2.28) for $n = N+1$,

$$1 - \eta(\varepsilon^{-\beta}(y-x)) \neq 0 \Longrightarrow |\varphi(\varepsilon^{-1}(y-x))| \leq c\frac{\|\Phi\|_{\mathcal{V}}}{(\rho\varepsilon^{\beta-1})^{N+1}}.$$

From (2.9) and the above estimate we infer

(2.33) $$|F_\varepsilon^1(x)| \leq c\varepsilon^{(1-\beta)(N+1)}\|\Phi\|_{\mathcal{V}}.$$

2. From (2.16) we derive

$$|B_\varepsilon(x)| \leq \varepsilon^{N-1} \int_{\mathbb{R}^{N-1}} (1 - \eta_\varepsilon(\varepsilon s))|\varphi(s,0)|ds.$$

In view of (2.10) we have

(2.34) $$\forall t \in \mathcal{O}, \qquad |y(t) - x| = \sqrt{|t|^2 + \psi(t)^2}.$$

Yet, using (2.11) and a Taylor–Lagrange expansion, we get

(2.35) $$\forall t \in \mathcal{O}, \qquad |\psi(t)| \leq c|t|^2.$$

Note that, by compactness of $\partial\Omega$, the above constant $c$ can be chosen independent of $x$. We deduce that

$$\forall t \in \mathcal{O}, \qquad |y(t) - x| \leq \sqrt{|t|^2 + c|t|^4} \leq \lambda|t|$$

for some $\lambda \geq 1$, and therefore

$$\forall t \in \mathcal{O}, \qquad |t| \leq \frac{\rho}{\lambda} \varepsilon^{\beta} \implies \varepsilon^{-\beta}|y(t) - x| \leq \rho \implies \eta_{\varepsilon}(t) = 1.$$

Set $\alpha = \rho/\lambda$, possibly decreased so that $B(0, \alpha) \subset \mathcal{O}$. Thus, for all $t \in \mathbb{R}^{N-1}$, $|t| \leq \alpha \varepsilon^{\beta}$ implies $\eta_{\varepsilon}(t) = 1$. Using also (2.28) for $n = N + 1$, we arrive at

(2.36)
$$|B_{\varepsilon}(x)| \leq c\varepsilon^{N-1}\|\Phi\|_{\mathcal{V}} \int_{\mathbb{R}^{N-1}\backslash B(0,\alpha\varepsilon^{\beta-1})} \frac{1}{|s|^{N+1}} ds = c\varepsilon^{N-1}\|\Phi\|_{\mathcal{V}} \left(\alpha^{-1}\varepsilon^{1-\beta}\right)^2$$

(2.37)    $\leq c\varepsilon^{N-1+2(1-\beta)}\|\Phi\|_{\mathcal{V}}.$

3. From (2.17), we obtain

$$|C_{\varepsilon}(x)| \leq \varepsilon^{N-1} \int_{\mathbb{R}^{N-1}} \eta_{\varepsilon}(\varepsilon s) \left|\varphi(s, \varepsilon^{-1}\psi(\varepsilon s)) - \varphi(s, 0)\right| ds.$$

The mean value inequality entails

$$\left|\varphi(s, \varepsilon^{-1}\psi(\varepsilon s)) - \varphi(s, 0)\right| \leq |\varepsilon^{-1}\psi(\varepsilon s)| \sup_{|t| \leq \varepsilon^{-1}|\psi(\varepsilon s)|} \left|\frac{\partial \varphi}{\partial z_N}(s, t)\right|.$$

From (2.32) with $n = N$ and (2.35) we derive

$$\left|\varphi(s, \varepsilon^{-1}\psi(\varepsilon s)) - \varphi(s, 0)\right| \leq c\|\Phi\|_{\mathcal{V}}\varepsilon|s|^{2-N}.$$

Yet, (2.34) yields $|y(t) - x| \geq |t|$ for all $t \in \mathcal{O}$, and hence

(2.38)        $\forall t \in \mathcal{O}, \qquad |t| \geq 2\rho\varepsilon^{\beta} \implies \varepsilon^{-\beta}|y(t) - x| \geq 2\rho \implies \eta_{\varepsilon}(t) = 0.$

We conclude that

(2.39)        $$|C_{\varepsilon}(x)| \leq c\|\Phi\|_{\mathcal{V}}\varepsilon^{N} \int_{B(0,2\rho\varepsilon^{\beta-1})} |s|^{2-N} ds = c\|\Phi\|_{\mathcal{V}}\varepsilon^{N+\beta-1}.$$

4. We get from (2.18) that

$$|D_{\varepsilon}(x)| \leq \varepsilon^{N-1} \int_{\mathbb{R}^{N-1}} \eta_{\varepsilon}(\varepsilon s)|\varphi(s, \varepsilon^{-1}\psi(\varepsilon s))||\nabla\psi(\varepsilon s)| ds.$$

Using that $|\nabla\psi(t)| \leq c|t|$ for all $t \in \mathcal{O}$, together with (2.28) for $n = N - 1$, we obtain

$$|D_{\varepsilon}(x)| \leq c\|\Phi\|_{\mathcal{V}}\varepsilon^{N} \int_{\mathbb{R}^{N-1}} \eta_{\varepsilon}(\varepsilon s)|s|^{2-N} ds.$$

Using now (2.38), we arrive at

(2.40)                               $|D_{\varepsilon}(x)| \leq c\|\Phi\|_{\mathcal{V}}\varepsilon^{N+\beta-1}.$

Gathering (2.22), (2.33), (2.36), (2.39), and (2.40), we finally obtain

(2.41)                       $|T_{\varepsilon}(\Phi) - \tilde{T}_{\varepsilon}(\Phi)| \leq c\|\Phi\|_{\mathcal{V}}\varepsilon^{\alpha} \qquad \forall \Phi \in \mathcal{C}_0^{\infty}(\mathbb{R})$

for the exponent

$$\alpha = 2 - N + \min((1-\beta)(N+1), N-1+2(1-\beta), N+\beta-1)$$
$$= \min(3 - \beta(N+1), 3 - 2\beta, 1 + \beta) = \min(3 - \beta(N+1), 1 + \beta).$$

This value is maximized when $3 - \beta(N+1) = 1 + \beta$, i.e., when $\beta = 2/(N+2)$. This corresponds to $\alpha = (N+4)/(N+2)$.

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**2.4. Extension to a function $\Phi \in \mathcal{V}$.** We shall now extend (2.41) to an arbitrary function $\Phi \in \mathcal{V}$. Expressing the integral in (2.2) in spherical coordinates entails

$$T_\varepsilon(\Phi) = \varepsilon^{-N} \int_0^\infty \int_{\mathcal{S}_{N-1}} \Phi(r) r^2 |\hat{u}(\varepsilon^{-1} rv)|^2 r^{N-1} d\sigma(v) dr = \int_0^\infty \Phi(r) r^2 w_\varepsilon(r) dr,$$

with

$$w_\varepsilon(r) = \varepsilon^{-N} r^{N-1} \int_{\mathcal{S}_{N-1}} |\hat{u}(\varepsilon^{-1} rv)|^2 dv,$$

and $\mathcal{S}_{N-1}$ is the unit sphere of $\mathbb{R}^{N-1}$. Note that, as $\hat{u} \in L^2(\mathbb{R}^N)$, we have $w_\varepsilon \in L^1(\mathbb{R}_+)$. We also write

$$\tilde{T}_\varepsilon(\Phi) = \int_0^\infty \Phi(r) r^2 \tilde{w}_\varepsilon(r) dr, \qquad \text{with} \qquad \tilde{w}_\varepsilon(r) = \frac{\varepsilon}{\pi} \frac{\mathrm{Per}(\Omega)}{r^2}.$$

Therefore we have

$$T_\varepsilon(\Phi) - \tilde{T}_\varepsilon(\Phi) = \int_0^\infty \Phi(r) r^2 [w_\varepsilon(r) - \tilde{w}_\varepsilon(r)] dr.$$

From (2.41) and the above equality we derive that

$$(2.42) \qquad \left| \int_0^\infty \Phi(r) r^2 [w_\varepsilon(r) - \tilde{w}_\varepsilon(r)] dr \right| \leq c\varepsilon^\alpha \|\Phi\|_{\mathcal{V}} \qquad \forall \Phi \in \mathcal{C}_0^\infty(\mathbb{R}).$$

We choose now an arbitrary function $\Phi \in \mathcal{V}$ and construct the sequence of auxiliary functions

$$(2.43) \qquad \Phi_n(r) = \Phi(r) \eta\left(\frac{r}{n}\right),$$

where $\eta \in \mathcal{C}^\infty(\mathbb{R})$ is such that $\eta(r) = 1$ if $|r| \leq 1$, $0 \leq \eta(r) \leq 1$ if $1 \leq |r| \leq 2$, and $\eta(r) = 0$ if $|r| \geq 2$. The differentiation of (2.43) at the order $k$ by the Leibniz formula and a reordering gives

$$\forall n \in \mathbb{N}^\star, \forall r \in \mathbb{R},$$

$$r^{k-2}(1+r^2)^2 \Phi_n^{(k)}(r) = \sum_{p=0}^k \binom{k}{p} r^{p-2}(1+r^2)^2 \Phi^{(p)}(r) \left(\frac{r}{n}\right)^{k-p} \eta^{(k-p)}\left(\frac{r}{n}\right).$$

For each $q \in \mathbb{N}$ the function $t \mapsto t^q \eta^{(q)}(t)$ belongs to $\mathcal{C}_0^\infty(\mathbb{R})$, and hence it is bounded. This entails

$$\forall n \in \mathbb{N}^\star, \qquad \|\Phi_n\|_{\mathcal{V}_k} \leq c_k \sum_{p=0}^k \|\Phi\|_{\mathcal{V}_p}$$

for some constants $c_k$ independent of $n$, and subsequently,

$$(2.44) \qquad \forall n \in \mathbb{N}^\star, \qquad \|\Phi_n\|_{\mathcal{V}} \leq c\|\Phi\|_{\mathcal{V}}.$$

Applying (2.42) to the function $\Phi_n$ and using (2.44), it follows that

$$\forall n \in \mathbb{N}^\star, \qquad \left| \int_0^\infty \eta\left(\frac{r}{n}\right) \Phi(r) r^2 [w_\varepsilon(r) - \tilde{w}_\varepsilon(r)] dr \right| \le c\varepsilon^\alpha \|\Phi\|_\mathcal{V}.$$

By Lebesgue's dominated convergence theorem, we can pass to the limit and find

$$\left| \int_0^\infty \Phi(r) r^2 [w_\varepsilon(r) - \tilde{w}_\varepsilon(r)] dr \right| \le c\varepsilon^\alpha \|\Phi\|_\mathcal{V},$$

that is,

$$|T_\varepsilon(\Phi) - \tilde{T}_\varepsilon(\Phi)| \le c\varepsilon^\alpha \|\Phi\|_\mathcal{V}.$$

The proof of Theorem 2.1 is now complete.

**3. Extension to a boundary value problem.** We assume now that $\Omega \subset\subset D$, where $D$ is a bounded Lipschitz domain of $\mathbb{R}^N$ and $\Omega$ has a $\mathcal{C}^2$ boundary. We consider the following problem: find $v_\varepsilon \in H^1(D)$ such that

$$(3.1) \qquad \begin{cases} -\varepsilon^2 \Delta v_\varepsilon + v_\varepsilon = u & \text{in} \quad D, \\ \partial_n v_\varepsilon = 0 & \text{on} \quad \partial D, \end{cases}$$

with $u$ the characteristic function of $\Omega$ in $D$, and set

$$(3.2) \qquad E_\varepsilon(\Omega) = \|u - v_\varepsilon\|_{L^2(D)}^2.$$

Note that we have restricted ourselves to the case $m = 1$ merely for simplicity. We shall show that $E_\varepsilon(\Omega)$ obeys the same first order asymptotic expansion as in the unbounded case.

THEOREM 3.1. *The following asymptotic expansion holds when $\varepsilon$ goes to zero:*

$$(3.3) \qquad E_\varepsilon(\Omega) = \frac{\varepsilon}{4} \operatorname{Per}(\Omega) + O(\varepsilon^{\frac{N+4}{N+2}}).$$

*Proof.* We make the splitting $v_\varepsilon = u_\varepsilon + e_\varepsilon$ with $u_\varepsilon \in H^1(\mathbb{R}^N)$ and $e_\varepsilon \in H^1(D)$, respectively, solutions of

$$(3.4) \qquad -\varepsilon^2 \Delta u_\varepsilon + u_\varepsilon = u \text{ in } \mathbb{R}^N, \qquad \begin{cases} -\varepsilon^2 \Delta e_\varepsilon + e_\varepsilon = 0 & \text{in} \quad D, \\ \partial_n e_\varepsilon = -\partial_n u_\varepsilon & \text{on} \quad \partial D. \end{cases}$$

Above, $u$ is extended by zero outside $D$. We introduce the rescaled function $U_\varepsilon(x) := u_\varepsilon(\varepsilon x)$, which solves

$$-\Delta U_\varepsilon + U_\varepsilon = u(\varepsilon x) \text{ in } \mathbb{R}^N.$$

Thus we can write for all $x \in \mathbb{R}^N$

$$U_\varepsilon(x) = \int_{\mathbb{R}^N} u(\varepsilon y) \Gamma(x - y) dy,$$

where $\Gamma$ is the fundamental solution of the operator $-\Delta + I$ in $\mathbb{R}^N$. By a change of variables we obtain

$$u_\varepsilon(x) = \varepsilon^{-N} \int_\Omega \Gamma\left(\frac{x - z}{\varepsilon}\right) dz.$$

Assume now that $\mathrm{dist}(x,\Omega) \geq \rho > 0$. By the Fourier transform, we can easily show that $|\Gamma(x)| = O(|x|^{-p})$ for all $p > 0$. This implies

$$\forall z \in \Omega, \qquad \left|\Gamma\left(\frac{x-z}{\varepsilon}\right)\right| \leq c\left(\frac{\varepsilon}{\rho}\right)^p.$$

We arrive at

$$|u_\varepsilon(x)| \leq c|\Omega|\rho^{-p}\varepsilon^{p-N}.$$

Similar estimates hold for $|\nabla u_\varepsilon(x)|$ and $|\Delta u_\varepsilon(x)|$, which provide, for any $k > 0$,

$$\text{(3.5)} \qquad \|u_\varepsilon\|_{H^1(\mathbb{R}^N \setminus D)} \leq c\varepsilon^k, \qquad \|\partial_n u_\varepsilon\|_{H^{-1/2}(\partial D)} \leq c\varepsilon^k.$$

Now, the variational formulation of (3.4) yields

$$\int_D (\varepsilon^2|\nabla e_\varepsilon|^2 + |e_\varepsilon|^2)dx = -\int_{\partial D} \partial_n u_\varepsilon e_\varepsilon dx,$$

from which we deduce

$$\text{(3.6)} \qquad \varepsilon^2\|e_\varepsilon\|_{H^1(D)} \leq c\|\partial_n u_\varepsilon\|_{H^{-1/2}(\partial D)} \leq c\varepsilon^k.$$

Then we write

$$\text{(3.7)} \quad E_\varepsilon(\Omega) = \|u - u_\varepsilon\|_{L^2(D)}^2 - 2\int_D e_\varepsilon(u - u_\varepsilon)dx + \|e_\varepsilon\|_{L^2(D)}^2$$

$$\text{(3.8)} \qquad = \|u - u_\varepsilon\|_{L^2(\mathbb{R}^N)}^2 - \|u_\varepsilon\|_{L^2(\mathbb{R}^N \setminus D)}^2 - 2\int_D e_\varepsilon(u - u_\varepsilon)dx + \|e_\varepsilon\|_{L^2(D)}^2.$$

By Theorem 1.1 we have

$$\text{(3.9)} \qquad \|u - u_\varepsilon\|_{L^2(\mathbb{R}^N)}^2 = \frac{\varepsilon}{4}\mathrm{Per}(\Omega) + O(\varepsilon^{\frac{N+4}{N+2}}).$$

Combining (3.8), (3.5), (3.6), and (3.9), using the Cauchy–Schwarz inequality, and choosing $k$ sufficiently large yields (3.3). $\square$

It is also of interest for the applications to study domains of the form $D\setminus\overline{\Omega}$, where $\Omega$ is defined as before. The peculiarity of this set is to touch the external boundary $\partial D$. The corresponding functional $E_\varepsilon(D\setminus\overline{\Omega})$ is defined by (3.1) and (3.2), with $u$ the characteristic function of $D\setminus\overline{\Omega}$. It turns out that the previous asymptotic expansion remains valid in this case, as stated in the following corollary.

COROLLARY 3.2. *The following asymptotic expansion holds when $\varepsilon$ goes to zero:*

$$\text{(3.10)} \qquad E_\varepsilon(D\setminus\overline{\Omega}) = \frac{\varepsilon}{4}\mathrm{Per}(\Omega) + O(\varepsilon^{\frac{N+4}{N+2}}).$$

*Proof.* We have by definition

$$E_\varepsilon(D\setminus\overline{\Omega}) = \|u^{D\setminus\overline{\Omega}} - v_\varepsilon^{D\setminus\overline{\Omega}}\|_{L^2(D)}^2,$$

where $u^{D\setminus\overline{\Omega}}$ is the characteristic function of $D\setminus\overline{\Omega}$, and $v_\varepsilon^{D\setminus\overline{\Omega}}$ solves

$$\begin{cases} -\varepsilon^2\Delta v_\varepsilon^{D\setminus\overline{\Omega}} + v_\varepsilon^{D\setminus\overline{\Omega}} = u^{D\setminus\overline{\Omega}} & \text{in} \quad D, \\ \partial_n v_\varepsilon^{D\setminus\overline{\Omega}} = 0 & \text{on} \quad \partial D. \end{cases}$$

Since $u^{D\setminus\overline{\Omega}} = 1 - u^{\Omega}$ (almost everywhere in $D$), with $u^{\Omega}$ the characteristic function of $\Omega$, and, by uniqueness, $v_{\varepsilon}^{D\setminus\overline{\Omega}} = 1 - v_{\varepsilon}^{\Omega}$, with $v_{\varepsilon}^{\Omega}$ the solution of (3.1) for $u = u^{\Omega}$, we derive

$$E_{\varepsilon}(D \setminus \overline{\Omega}) = \|u^{\Omega} - v_{\varepsilon}^{\Omega}\|_{L^2(D)}^2 = E_{\varepsilon}(\Omega).$$

Then we apply Theorem 3.1. $\square$

Note that, in this case, it is still the perimeter of $\Omega$ which is involved, not that of $D \setminus \overline{\Omega}$. In fact, this corresponds to the relative perimeter of $D \setminus \overline{\Omega}$ in $D$, namely $\sigma(\partial(D \setminus \overline{\Omega}) \cap D)$; see, e.g., [16].

**4. Topological sensitivity of the regularized perimeter.** We place ourselves in the context of section 3; i.e., we consider a bounded Lipschitz domain $D$ of $\mathbb{R}^N$ which will stand for "hold all." In this section we assume that $\varepsilon > 0$ is fixed. For all $u \in L^2(D)$, we denote by $L_{\varepsilon}u$ the solution $v_{\varepsilon}$ of (3.1), and we set

$$P_{\varepsilon}(u) = \int_D L_{\varepsilon}u(L_{\varepsilon}u - 2u)dx.$$

The functional $E_{\varepsilon}(\Omega)$ introduced in the previous section is defined for any measurable subset $\Omega$ of $D$ (here we do not need to assume further regularity or that $\Omega \subset\subset D$) by $E_{\varepsilon}(\Omega) = \|L_{\varepsilon}\chi_{\Omega} - \chi_{\Omega}\|_{L^2(D)}^2$, with $\chi_{\Omega}$ the characteristic function of $\Omega$ in $D$. Then the regularized perimeter $\text{Per}_{\varepsilon}(\Omega)$ defined by (1.2) satisfies

$$(4.1) \qquad \text{Per}_{\varepsilon}(\Omega) = \frac{4}{\varepsilon}\|L_{\varepsilon}\chi_{\Omega} - \chi_{\Omega}\|_{L^2(D)}^2 = \frac{4}{\varepsilon}\left[P_{\varepsilon}(\chi_{\Omega}) + |\Omega|\right],$$

where $|\Omega|$ is the $N$-dimensional Lebesgue measure of $\Omega$.

LEMMA 4.1. *For any $q \in ]1, 2]$ if $N = 2$, $q \in [6/5, 2]$ if $N = 3$, the functional $u \in L^q(D) \mapsto P_{\varepsilon}(u)$ is of class $\mathcal{C}^{\infty}$ in the sense of Fréchet. Its derivative in the direction $h \in L^q(D)$ is given by*

$$(4.2) \qquad DP_{\varepsilon}(u)h = 2\int_D (p_{\varepsilon} - v_{\varepsilon})hdx,$$

*where $v_{\varepsilon} = L_{\varepsilon}u$ is the direct state and $p_{\varepsilon}$ is an adjoint state solution of*

$$(4.3) \qquad \begin{cases} -\varepsilon^2\Delta p_{\varepsilon} + p_{\varepsilon} = v_{\varepsilon} - u & in \quad D, \\ \partial_n p_{\varepsilon} = 0 & on \quad \partial D. \end{cases}$$

*Proof.* First, by application of the Lax–Milgram theorem, the map $L_{\varepsilon} : (H^1(D))' \to H^1(D)$ is linear and continuous. In addition, we have the continuous embeddings $H^1(D) \hookrightarrow L^{q'}(D)$ and $L^q(D) \hookrightarrow (H^1(D))'$, where $q'$ is such that $1/q + 1/q' = 1$. Thus the map $u \in L^q(D) \mapsto P_{\varepsilon}(u)$ is of class $\mathcal{C}^{\infty}$ by composition. The standard rules of differential calculus provide

$$DP_{\varepsilon}(u)h = \int_D \left[L_{\varepsilon}h(L_{\varepsilon}u - 2u) + L_{\varepsilon}u(L_{\varepsilon}h - 2h)\right]dx.$$

A rearrangement and the replacement of $L_{\varepsilon}u$ by $v_{\varepsilon}$ yields

$$DP_{\varepsilon}(u)h = 2\int_D \left[(v_{\varepsilon} - u)L_{\varepsilon}h - v_{\varepsilon}h\right]dx.$$

Since the operator $L_\varepsilon : L^q(D) \to L^{q'}(D)$ is self-adjoint, we can also write

$$DP_\varepsilon(u)h = 2 \int_D [L_\varepsilon(v_\varepsilon - u)h - v_\varepsilon h]\, dx.$$

The definition of the adjoint state as $p_\varepsilon = L_\varepsilon(v_\varepsilon - u)$ leads to (4.2).  $\square$

THEOREM 4.2. *Let $\Omega$ be a measurable subset of $D$, and let $v_\varepsilon$, $p_\varepsilon$ be the direct and adjoint states, respectively, which are the solutions of*

$$\begin{cases} -\varepsilon^2 \Delta v_\varepsilon + v_\varepsilon = \chi_\Omega & in \quad D, \\ \partial_n v_\varepsilon = 0 & on \quad \partial D, \end{cases} \qquad \begin{cases} -\varepsilon^2 \Delta p_\varepsilon + p_\varepsilon = v_\varepsilon - \chi_\Omega & in \quad D, \\ \partial_n p_\varepsilon = 0 & on \quad \partial D. \end{cases}$$

*For any $q$ chosen as in Lemma 4.1 and any measurable subset $\tilde\Omega$ of $D$, we have*

$$(4.4) \qquad Per_\varepsilon(\tilde\Omega) - Per_\varepsilon(\Omega) = \int_D Per'_\varepsilon(\Omega)(\chi_{\tilde\Omega} - \chi_\Omega)dx + O(\|\chi_{\tilde\Omega} - \chi_\Omega\|^{2/q}_{L^1(D)}),$$

*with the function $Per'_\varepsilon(\Omega)$ given by*

$$Per'_\varepsilon(\Omega) = \frac{4}{\varepsilon}\left[1 + 2(p_\varepsilon - v_\varepsilon)\right].$$

*Proof.* We get from (4.1) that

$$\mathrm{Per}_\varepsilon(\tilde\Omega) - \mathrm{Per}_\varepsilon(\Omega) = \frac{4}{\varepsilon}\left[P_\varepsilon(\chi_{\tilde\Omega}) - P_\varepsilon(\chi_\Omega) + |\tilde\Omega| - |\Omega|\right].$$

A Taylor–Lagrange expansion of $P_\varepsilon$ yields

$$\mathrm{Per}_\varepsilon(\tilde\Omega) - \mathrm{Per}_\varepsilon(\Omega) = \frac{4}{\varepsilon}\left[DP_\varepsilon(\chi_\Omega)(\chi_{\tilde\Omega} - \chi_\Omega) + O(\|\chi_{\tilde\Omega} - \chi_\Omega\|^2_{L^q(D)}) + |\tilde\Omega| - |\Omega|\right].$$

Then (4.2) entails

$$\mathrm{Per}_\varepsilon(\tilde\Omega) - \mathrm{Per}_\varepsilon(\Omega)$$
$$= \frac{4}{\varepsilon}\left[\int_D 2(p_\varepsilon - v_\varepsilon)(\chi_{\tilde\Omega} - \chi_\Omega)dx + O(\|\chi_{\tilde\Omega} - \chi_\Omega\|^{2/q}_{L^1(D)}) + \int_D (\chi_{\tilde\Omega} - \chi_\Omega)dx\right].$$

A rearrangement completes the proof.  $\square$

We straightforwardly infer the following behavior regarding a perturbation localized around a point, for instance, ball-shaped.

COROLLARY 4.3. *Suppose that $\chi_{\tilde\Omega} - \chi_\Omega = s\chi_{B(z,\rho)}$ for some $z \in D \setminus \partial\Omega$, $\rho > 0$, and $s = -1$ if $z \in \Omega$, $s = 1$ if $z \in D \setminus \overline\Omega$. Then we have when $\rho \to 0$*

$$(4.5) \qquad Per_\varepsilon(\tilde\Omega) - Per_\varepsilon(\Omega) = s|B(z,\rho)|Per'_\varepsilon(\Omega)(z) + o(|B(z,\rho)|).$$

*Proof.* By elliptic regularity, $Per'_\varepsilon(\Omega)$ is continuous in the vicinity of $z$, hence, as $\rho \to 0$, the first term on the right-hand side of (4.4) is equal to $sPer'_\varepsilon(\Omega)(z)|B(z,\rho)| + o(|B(z,\rho)|)$. The second term is an $O(|B(z,\rho)|^{2/q})$ which, by choosing $q < 2$, is an $o(|B(z,\rho)|)$.  $\square$

In view of (4.5), the function $Per'_\varepsilon(\Omega)$ can be identified as the topological derivative [5, 13, 21, 22, 23] of the shape functional $Per_\varepsilon$ evaluated at $\Omega$. In particular, we see that the functional $Per_\varepsilon$ behaves like the volume of the perturbation, in contrast to the perimeter.

**5. Application to topology optimization.** We address topology optimization problems of the form

$$(5.1) \qquad \min_{\Omega \subset D} J(\Omega) := I(\Omega) + \alpha \operatorname{Per}(\Omega).$$

Above, $I(\Omega)$ is a given shape functional, $\alpha$ is a positive parameter, and $\operatorname{Per}(\Omega)$ stands for the relative perimeter of $\Omega$ in $D$, whose definition can be extended to any measurable subset of $D$ [9, 11, 16], including the value $+\infty$ for sets of nonfinite perimeter. To fix our ideas and introduce the method we focus on the functional

$$(5.2) \qquad I(\Omega) = \frac{\beta}{2}\|A\chi_\Omega - f\|^2_{L^2(D)} + \int_\Omega w\,dx,$$

with the data $\beta \geq 0$, $w \in L^1(D)$, $f \in L^2(D)$, and $A : u \mapsto y = Au$ the solution operator to the model problem

$$(5.3) \qquad \begin{cases} -\Delta y = u & \text{in} \quad D, \\ y = 0 & \text{on} \quad \partial D. \end{cases}$$

The first term in (5.2) represents a quadratic misfitting to some measurement or target $f$, while the second term is a (possibly weighted) volume penalization. Since $u$ is searched in the class of characteristic functions, the volume penalization can be seen as an $L^1$ control cost, and $u$ is a particular form of sparse optimal control; see, e.g., [19]. The existence of minimizers for (5.1) is ensured by standard arguments (see, e.g., Theorem 4.1.4 of [16]), based on the compact embedding of $BV(D)$ into $L^1(D)$. For every $\varepsilon > 0$ we define the approximate problem

$$(5.4) \qquad \min_{\Omega \subset D} J_\varepsilon(\Omega) := I(\Omega) + \alpha\operatorname{Per}_\varepsilon(\Omega).$$

The existence of minimizers for (5.4) is proved in Appendix B under the condition

$$(5.5) \qquad \beta\varepsilon C_P^2(\varepsilon^2 + C_P^2) < 4\alpha,$$

where $C_P$ is the Poincaré constant on $H_0^1(D)$, which holds true as soon as $\varepsilon$ is small enough. We use the continuation method described below.

ALGORITHM 1.
1. *Define an initial domain $\Omega_0$ and a decreasing sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of positive numbers such that $\lim_{n\to\infty} \varepsilon_n = 0$. Set $n = 0$.*
2. *Solve (5.4) with $\varepsilon = \varepsilon_n$ and the initial guess $\Omega_n$. Call $\Omega_{n+1}$ the obtained solution.*
3. *Increment $n \leftarrow n+1$ and goto step (2).*

Note that in the beginning of the iterations, when $\varepsilon$ is large, in particular when the existence of minimizers of (5.4) may fail, one can be content with an approximate minimizer.

To solve (5.4), we use the algorithm introduced in [7] and analyzed in [6]. We recall its main features. First, we need the topological derivative of the functional $J_\varepsilon$. We know from Theorem 4.2 the topological derivative of $\operatorname{Per}_\varepsilon$. The topological derivative of $I$, which is obtained in a similar manner (see also [5, 6]), reads

$$I'(\Omega) = \beta A(A\chi_\Omega - f) + w.$$

We get the topological asymptotic expansion

$$(5.6) \qquad J_\varepsilon(\tilde\Omega) - J_\varepsilon(\Omega) = \int_D J'_\varepsilon(\Omega)(\chi_{\tilde\Omega} - \chi_\Omega)dx + o(\|\chi_{\tilde\Omega} - \chi_\Omega\|_{L^1(D)}),$$

with

$$J'_\varepsilon(\Omega) = I'(\Omega) + \alpha \mathrm{Per}'_\varepsilon(\Omega).$$

From (5.6) we deduce the following necessary optimality conditions [6]:

$$(5.7) \qquad \begin{array}{ll} J'_\varepsilon(\Omega) \le 0 & \text{a.e. in} \quad \Omega, \\ J'_\varepsilon(\Omega) \ge 0 & \text{a.e. in} \quad D \setminus \overline\Omega. \end{array}$$

To solve these conditions, we represent every domain $\Omega \subset D$ by a so-called level-set function $\psi : D \to \mathbb{R}$ constructed so that

$$\Omega = \Omega(\psi) := \{x \in D, \psi(x) < 0\}.$$

We equip the set of real valued functions defined on $D$ with the equivalence relation

$$\psi_1 \sim \psi_2 \iff \exists \mu > 0, \qquad \psi_1 = \mu\psi_2.$$

Therefore, the conditions (5.7) will be satisfied by the domain $\Omega(\psi)$ whenever

$$(5.8) \qquad J'_\varepsilon(\Omega(\psi)) \sim \psi.$$

We solve this equation by the fixed point iteration with relaxation applied to the equivalence classes, namely,

$$(5.9) \qquad \psi_{k+1} \sim (1 - \lambda_k)\psi_k + \lambda_k J'_\varepsilon(\Omega(\psi_k)),$$

for some $\lambda_k \in ]0,1]$. In fact, it turns out to be convenient to handle representatives on the unit sphere $\mathcal{S}$ of some Hilbert space $\mathcal{H}$ of functions on $D$, for instance, $\mathcal{H} = L^2(D)$. Denoting by $\langle ., . \rangle_\mathcal{H}$ and $\|.\|_\mathcal{H}$ the inner product and the norm of $\mathcal{H}$, respectively, (5.9) can be reformulated as

$$(5.10) \qquad \psi_{k+1} = \frac{1}{\sin\theta_k}\left[\sin((1-\tau_k)\theta_k)\psi_k + \sin(\tau_k\theta_k)\frac{J'_\varepsilon(\Omega(\psi_k))}{\|J'_\varepsilon(\Omega(\psi_k))\|_\mathcal{H}}\right],$$

with the angle

$$\theta_k = \arccos\left\langle \psi_k, \frac{J'_\varepsilon(\Omega(\psi_k))}{\|J'_\varepsilon(\Omega(\psi_k))\|_\mathcal{H}} \right\rangle_\mathcal{H},$$

and $\tau_k \in ]0,1]$ acting as the stepsize in place of $\lambda_k$. Note that the angle $\theta_k$ has some meaning, since solving (5.8) amounts to driving $\theta_k$ to zero. We arrive at the following algorithm.

ALGORITHM 2.
1. *Choose an initial function $\psi_0 \in \mathcal{S}$. Set $k = 0$.*
2. *Determine $\psi_{k+1} \in \mathcal{S}$ by* (5.10)*, with $\tau_k \in ]0,1]$ chosen such that*

$$J_\varepsilon(\Omega(\psi_{k+1})) \le J_\varepsilon(\Omega(\psi_k)).$$

3. *Increment $n \leftarrow n + 1$ and goto step* (2).

In the implementation, $\tau_k$ is determined by a line search of Armijo type (see [6]). For the functional (5.2), the convergence (up to a subsequence) of Algorithm 2 to a local minimizer of (5.4) is proved in [6] under some regularity assumptions on the sequence of functions $\psi_k$.

FIG. 1. *Basic case* 1: $\alpha = 0$ *(left)*, $\alpha = 0.1$ *(middle)*, $\alpha = 0.2$ *(right)*.

**6. Numerical experiments.** In the following examples the spatial dimension is $N = 2$. We choose the full domain initialization $\Omega_0 = D$, more precisely, $\psi_0 = -1/\|1\|_{\mathcal{H}}$ with $\mathcal{H} = L^2(D)$. All the partial differential equations involved are solved by piecewise linear finite elements on a structured triangular mesh. The sequence of regularization parameters is chosen as $\varepsilon_n = 1/2^n$, and 15 iterations of Algorithm 1 are performed. Actually, we observe that almost no more evolution occurs when $\varepsilon_n$ becomes smaller than the mesh size. The stopping criterion of Algorithm 2 is $\theta_k \leq 0.1°$. This value ensures that (5.7) is satisfied with a good accuracy.

In subsections 6.1 and 6.2 we consider an objective functional of form (5.2), starting from the elementary case where $\beta = 0$. Then, to illustrate the wider range of applicability of the method, we address in subsection 6.3 problems related to the optimal design of microstructures.

**6.1. Basic cases: Weighted volume minimization.** For this first series of tests the coefficient $\beta$ in (5.2) is set to zero, and thus the state equation (5.3) is not involved. The hold all $D$ is the unit square $]0, 1[^2$. We use a mesh with 51521 nodes.

**6.1.1. Example 1.** The function $w$ is chosen as

$$w(x_1, x_2) = \begin{cases} -1 & \text{if } 0.2 \leq x_1, x_2 \leq 0.8, \\ 1 & \text{otherwise.} \end{cases}$$

In Figure 1 we present the results obtained with the coefficients $\alpha = 0$, $\alpha = 0.1$, and $\alpha = 0.2$. Of course, for $\alpha = 0$, the optimal solution is the rectangle $]0.2, 0.8[^2$. For $\alpha > 0$, the contribution of the perimeter term is highlighted by the rounded corners, obviously yielding a decrease of the perimeter.

**6.1.2. Example 2.** In order to demonstrate the ability of the algorithm to deal with topology changes and illustrate Corollary 3.2, we now choose

$$w(x_1, x_2) = \begin{cases} 1 & \text{if } 0.2 \leq x_1, x_2 \leq 0.8, \\ -1 & \text{otherwise.} \end{cases}$$

The results obtained for the same values of $\alpha$ as in Example 1 are shown in Figure 2.

**6.1.3. Example 3.** The purpose of this example is to show that the proposed algorithm can also be used when there exist junction points between $\partial\Omega$ and $\partial D$, although this case has not been treated in the theory. In fact, if the junctions occur at right angles, it is intuitively clear that, due to the Neumann boundary condition in (3.1), the functional $\text{Per}_\varepsilon(\Omega)$ still approximates the relative perimeter of $\Omega$ in $D$, namely, $\sigma(\partial\Omega \cap D)$. We consider the function

$$w(x_1, x_2) = \sin(2\pi x_1)\sin(2\pi x_2).$$

Figure 3 shows the results obtained with the coefficients $\alpha = 0$, $\alpha = 0.01$, and $\alpha = 0.1$.

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FIG. 2. *Basic case 2: $\alpha = 0$ (left), $\alpha = 0.1$ (middle), $\alpha = 0.2$ (right).*



FIG. 3. *Basic case 3: $\alpha = 0$ (left), $\alpha = 0.01$ (middle), $\alpha = 0.1$ (right).*

**6.1.4. Example 4.** In the last example of this series we combine topology changes and junctions of boundaries by choosing

$$w(x_1, x_2) = \sin(6\pi x_1)\sin(6\pi x_2).$$

The results obtained with the coefficients $\alpha = 0$, $\alpha = 0.01$, and $\alpha = 0.02$ are depicted in Figure 4. We remark here that the perimeter penalization modifies the topology of the solution.

**6.2. A least squares problem for the Poisson equation.** We now choose $\beta = 1$, $w \equiv 10^{-3}$, $D =]-1,1[\times]0,1[$, and $f \equiv 0.05$. The problem can be interpreted as finding the optimal location of a distributed heat source in order to approximate a given temperature field, with penalizations of the surface and the perimeter of the source region. The results obtained with selected values of $\alpha$, using a mesh with 57961 nodes, are displayed in Figure 5.

**6.3. Topology optimization of microstructures.** We consider the problem of optimizing the distribution of a two-phase elastic material within a periodic microstructure in order to obtain the desired macroscopic properties. The macroscopic elasticity tensor $C_\Omega$ at a given point of the structure is obtained by a standard model of homogenization involving the solutions of canonical linear elasticity problems defined over a representative volume element (RVE). The objective functional is of form (5.1), where $D$ is the RVE, $\Omega$ is the region occupied by one of the phases, say the stiffest one, and

$$I(\Omega) = h(C_\Omega) + \ell|\Omega|.$$

Here, $h(C_\Omega)$ is the targeted mechanical quantity and $\ell > 0$ is a fixed Lagrange multiplier associated with a volume constraint. This problem has been treated without perimeter penalization in [8], to which we refer the reader for details on the model and the topological sensitivity analysis. Perimeter penalization is a fairly standard way to
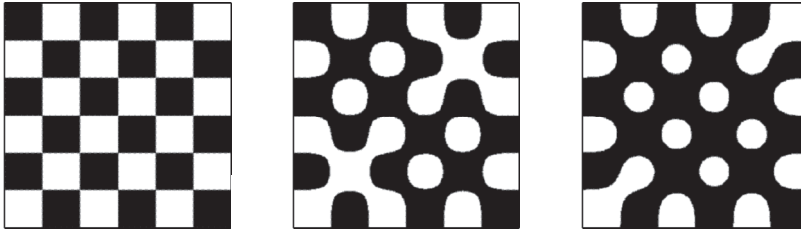
FIG. 4. *Basic case 4: $\alpha = 0$ (left), $\alpha = 0.01$ (middle), $\alpha = 0.02$ (right).*



FIG. 5. *Least squares problem: $\alpha = 10^{-6}$ (left), $\alpha = 10^{-5}$ (middle), and $\alpha = 10^{-4}$ (right).*

incorporate the notion of manufacturing complexity in the formulation of structural optimal design problems. Its implementation in the context of topology optimization is usually done by heuristic techniques (see, e.g., [10]).

For the problem under consideration, the convergence analysis of [6] does not apply; in particular, there is no guarantee of convergence at the discrete level. It is therefore important to use sufficiently fine meshes. In order to reduce the computational effort, a coarse-to-fine mesh technique is used for the first value of $\varepsilon$, which is the most time consuming since the initial guess may be far from the solution. Specifically, a uniform mesh refinement is performed each time the stepsize $\tau_k$ becomes less than 0.01. Overall, two such refinements are performed to reach the final mesh. In the subsequent steps of the algorithm, $\varepsilon$ is updated either when $\theta_k < 0.1°$ or when $\tau_k < 0.01$.

In the following examples the RVE is the square $]0, 1[^2$, and plane stress conditions are assumed. The Young modulus is set to 1 in $\Omega$ and 0.01 in $D \setminus \Omega$. The Poisson ratio of both phases is 0.3. The computer time of the whole procedure for each example is about 10 minutes on a standard PC for a final mesh containing 51521 nodes. Of course this time could be reduced by choosing less stringent stopping criteria.

**6.3.1. Bulk modulus maximization.** The homogenized bulk modulus $K_\Omega$ measures the stiffness under uniform compression. We define $h(C_\Omega)$ as the inverse of this quantity, namely,

$$h(C_\Omega) = \frac{1}{K_\Omega} = (C_\Omega^{-1})_{1111} + 2(C_\Omega^{-1})_{1122} + (C_\Omega^{-1})_{2222}.$$

The results obtained for $\ell = 20$ and different values of $\alpha$ are shown in Figure 6.

**6.3.2. Poisson ratio minimization.** General orthotropic materials admit two Poisson ratios $\nu_\Omega^{12}$ and $\nu_\Omega^{21}$ corresponding to the principal directions. In order to treat the two directions equally we define $h(C_\Omega)$ as the sum of the two homogenized Poisson ratios, namely,

$$h(C_\Omega) = \nu_\Omega^{12} + \nu_\Omega^{21} = -\frac{(C_\Omega^{-1})_{1122}}{(C_\Omega^{-1})_{1111}} - \frac{(C_\Omega^{-1})_{1122}}{(C_\Omega^{-1})_{2222}}.$$
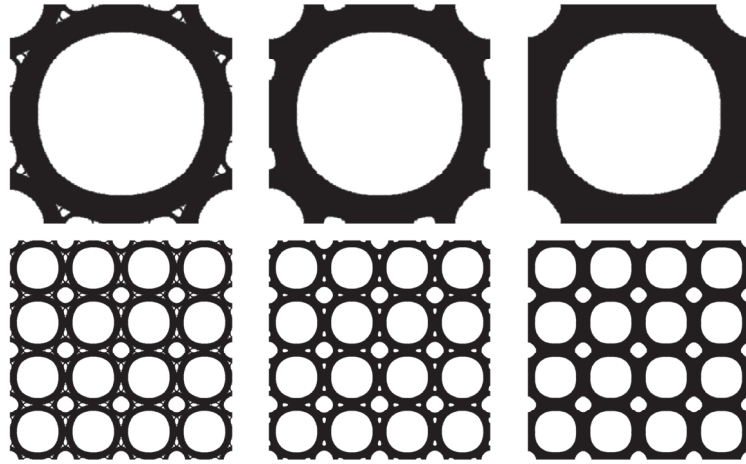
Fig. 6. *Bulk modulus maximization: $\alpha = 0$ (left), $\alpha = 0.1$ (middle), $\alpha = 0.5$ (right). The optimized RVE and the associated periodic microstructure are shown in the top and bottom row, respectively.*



Fig. 7. *Poisson ratio minimization: RVE for $\alpha = 0$ (left), $\alpha = 0.01$ (middle), and $\alpha = 0.02$ (right). The corresponding Poisson ratios are $-0.345$, $-0.319$, and $-0.260$, respectively.*
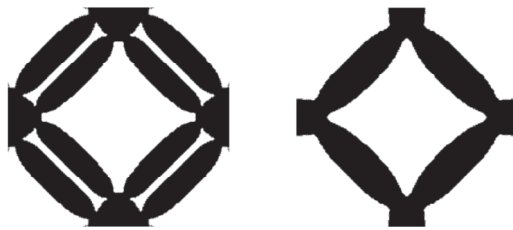


Fig. 8. *Poisson ratio maximization: RVE for $\alpha = 0$ (left) and $\alpha = 0.1$ (right). The corresponding Poisson ratios are $0.871$ and $0.831$, respectively.*

We do not impose any volume constraint ($\ell = 0$). Our findings are shown in Figure 7.

**6.3.3. Poisson ratio maximization.** We now choose $h(C_\Omega) = -\nu_\Omega^{12} - \nu_\Omega^{21}$ and again $\ell = 0$. The corresponding results are reported in Figure 8.

**7. Conclusion.** A family of regularized perimeter functionals $\text{Per}_\varepsilon$ has been introduced in this paper. First, the convergence of $\text{Per}_\varepsilon$ to the exact perimeter when $\varepsilon \to 0$ was proved. Then, the topological sensitivity analysis of $\text{Per}_\varepsilon$ was carried out, leading to a numerical procedure dedicated to the topological optimization of

shape functionals involving a perimeter term in their constituents. This algorithm is illustrated by several numerical experiments in the contexts of sparse optimal control and structural optimal design. They show that the perimeter term can effectively be used to control the smoothness and complexity of the solution, a feature which is not only useful for guaranteeing the well-posedness of the problem but is also of interest in engineering applications in order to take into account manufacturing costs, for instance.

**Appendix A. An auxiliary calculus.** In this appendix we prove by induction the relation (2.25) for every $q \in \mathbb{N}$. Obviously it is true for $q = 0$. Suppose now that it is true for some $q \in \mathbb{N}$. The differentiation gives

$$\frac{\partial^{q+1}}{\partial \zeta_j^{q+1}}\left(\frac{\Phi(|\zeta|)}{|\zeta|^2}\right) = -(2q+2)\frac{\zeta_j}{|\zeta|^{2q+4}}\sum_{p=0}^{q}\Phi^{(p)}(|\zeta|)P_{p,q}(|\zeta|,\zeta_j)$$

$$+ \frac{1}{|\zeta|^{2q+2}}\sum_{p=0}^{q}\left[\Phi^{(p+1)}(|\zeta|)\frac{\zeta_j}{|\zeta|}P_{p,q}(|\zeta|,\zeta_j) + \Phi^{(p)}(|\zeta|)\right.$$

$$\left.\times\left(\partial_1 P_{p,q}(|\zeta|,\zeta_j)\frac{\zeta_j}{|\zeta|} + \partial_2 P_{p,q}(|\zeta|,\zeta_j)\right)\right].$$

For each $p \in \{0, \ldots, q\}$ we set

$$P_{p,q+1}^1(|\zeta|,\zeta_j) = -(2q+2)\zeta_j P_{p,q}(|\zeta|,\zeta_j),$$

$$P_{p+1,q+1}^2(|\zeta|,\zeta_j) = \zeta_j|\zeta|P_{p,q}(|\zeta|,\zeta_j),$$

$$P_{p,q+1}^3(|\zeta|,\zeta_j) = \partial_1 P_{p,q}(|\zeta|,\zeta_j)\zeta_j|\zeta| + \partial_2 P_{p,q}(|\zeta|,\zeta_j)|\zeta|^2.$$

We note that each polynomial $P_{\alpha,\beta}^l$ is homogeneous of degree $\alpha + \beta$. We obtain

$$\frac{\partial^{q+1}}{\partial \zeta_j^{q+1}}\left(\frac{\Phi(|\zeta|)}{|\zeta|^2}\right) = \frac{1}{|\zeta|^{2q+4}}\sum_{p=0}^{q}\left[\Phi^{(p)}(|\zeta|)P_{p,q+1}^1(|\zeta|,\zeta_j) + \Phi^{(p+1)}(|\zeta|)P_{p+1,q+1}^2(|\zeta|,\zeta_j)\right.$$

$$\left.+ \Phi^{(p)}(|\zeta|)P_{p,q+1}^3(|\zeta|,\zeta_j)\right].$$

A rearrangement entails

$$\frac{\partial^{q+1}}{\partial \zeta_j^{q+1}}\left(\frac{\Phi(|\zeta|)}{|\zeta|^2}\right) = \frac{1}{|\zeta|^{2q+4}}\sum_{p=0}^{q+1}\Phi^{(p)}(|\zeta|)\left[P_{p,q+1}^1(|\zeta|,\zeta_j) + P_{p,q+1}^2(|\zeta|,\zeta_j) + P_{p,q+1}^3(|\zeta|,\zeta_j)\right],$$

where the undefined polynomials $P_{q+1,q+1}^1$, $P_{0,q+1}^2$, and $P_{q+1,q+1}^3$ have been set to zero. It suffices now to set $P_{p,q+1} = P_{p,q+1}^1 + P_{p,q+1}^2 + P_{p,q+1}^3$ to complete the proof.

**Appendix B. Existence of minimizers for the regularized problem.** This appendix deals with the existence of minimizers for (5.4). We present a proof in three steps: first we relax the problem, then we show that the relaxed problem admits solutions, and finally we prove that these solutions are characteristic functions.

We define for all $u \in L^2(D)$

(B.1)          $\tilde{J}_\varepsilon(u) = \beta \|Au - f\|^2_{L^2(D)} + \langle w, u \rangle + \dfrac{4\alpha}{\varepsilon} \left[ \langle L_\varepsilon u, L_\varepsilon u - 2u \rangle + \langle 1, u \rangle \right],$

with $\langle .,. \rangle$ the inner product of $L^2(D)$. In view of (4.1) and (5.2) we have $J_\varepsilon(\Omega) = \tilde{J}_\varepsilon(\chi_\Omega)$ for each measurable subset $\Omega$ of $D$. By the direct method of calculus of variations and the weak compactness of $L^2(D, [0,1])$, we easily obtain the existence of minimizers of $\tilde{J}_\varepsilon$ in $L^2(D, [0,1])$. We now argue that $\tilde{J}_\varepsilon$ is strictly concave, so that its minimizers on $L^2(D, [0,1])$ are extreme points, i.e., characteristic functions. The quadratic form associated with (B.1) is given by

$$\langle Qu, u \rangle = \beta \|Au\|^2_{L^2(D)} + \frac{4\alpha}{\varepsilon} \langle L_\varepsilon u, L_\varepsilon u - 2u \rangle.$$

For simplicity we set $v_\varepsilon = L_\varepsilon u$ and $y = Au$, and hence

$$\langle Qu, u \rangle = \beta \|y\|^2_{L^2(D)} + \frac{4\alpha}{\varepsilon} (\|v_\varepsilon\|^2_{L^2(D)} - 2\langle v_\varepsilon, u \rangle).$$

From $\varepsilon^2 \|\nabla v_\varepsilon\|^2_{L^2(D)} + \|v_\varepsilon\|^2_{L^2(D)} = \langle v_\varepsilon, u \rangle$ we infer

$$\langle Qu, u \rangle \le \beta \|y\|^2_{L^2(D)} - \frac{4\alpha}{\varepsilon} \langle v_\varepsilon, u \rangle.$$

By ellipticity of $L_\varepsilon$ we have for any $\varphi \in H^1(D)$

$$\frac{1}{2}(\varepsilon^2 \|\nabla v_\varepsilon\|^2_{L^2(D)} + \|v_\varepsilon\|^2_{L^2(D)}) - \langle v_\varepsilon, u \rangle \le \frac{1}{2}(\varepsilon^2 \|\nabla \varphi\|^2_{L^2(D)} + \|\varphi\|^2_{L^2(D)}) - \langle \varphi, u \rangle.$$

Choosing $\varphi = ty$ for an arbitrary $t \in \mathbb{R}$ and using that $\langle y, u \rangle = \|\nabla y\|^2_{L^2(D)}$ we arrive at

$$-\frac{1}{2}\langle v_\varepsilon, u \rangle \le t \left[ \left( \frac{\varepsilon^2}{2} t - 1 \right) \|\nabla y\|^2_{L^2(D)} + \frac{t}{2} \|y\|^2_{L^2(D)} \right].$$

Next the Poincaré inequality and a rearrangement entail

$$t \left[ 1 - \frac{t}{2} \left( \varepsilon^2 + C_P^2 \right) \right] \|\nabla y\|^2_{L^2(D)} \le \frac{1}{2} \langle v_\varepsilon, u \rangle.$$

In order to maximize the polynomial in $t$ on the left-hand side we choose $t = 1/(\varepsilon^2 + C_P^2)$, which yields

$$\|\nabla y\|^2_{L^2(D)} \le (\varepsilon^2 + C_P^2)\langle v_\varepsilon, u \rangle.$$

Using again the Poincaré inequality we eventually arrive at

$$\langle Qu, u \rangle \le \left[ \beta C_P^2 (\varepsilon^2 + C_P^2) - \frac{4\alpha}{\varepsilon} \right] \langle L_\varepsilon u, u \rangle.$$

We deduce that $\tilde{J}_\varepsilon(u)$ is strictly concave under the condition (5.5).

## REFERENCES

[1] G. ALLAIRE, *Shape Optimization by the Homogenization Method*, Appl. Math. Sci. 146, Springer-Verlag, New York, 2002.

[2] G. ALLAIRE, *Conception optimale de structures*, Math. Appl. (Berlin) 58, Springer-Verlag, Berlin, 2007.

[3] G. ALLAIRE, F. JOUVE, AND A.-M. TOADER, *Structural optimization using sensitivity analysis and a level-set method*, J. Comput. Phys., 194 (2004), pp. 363–393.

[4] L. AMBROSIO AND G. BUTTAZZO, *An optimal design problem with perimeter penalization*, Calc. Var. Partial Differential Equations, 1 (1993), pp. 55–69.

[5] S. AMSTUTZ, *Sensitivity analysis with respect to a local perturbation of the material property*, Asymptot. Anal., 49 (2006), pp. 87–108.

[6] S. AMSTUTZ, *Analysis of a level-set method for topology optimization*, Optim. Methods Softw., 26 (2011), pp. 555–573.

[7] S. AMSTUTZ AND H. ANDRÄ, *A new algorithm for topology optimization using a level-set method*, J. Comput. Phys., 216 (2006), pp. 573–588.

[8] S. AMSTUTZ, S. M. GIUSTI, A. A. NOVOTNY, AND E. A. DE SOUZA NETO, *Topological derivative for multi-scale linear elasticity models applied to the synthesis of microstructures*, Internat. J. Numer. Methods Engrg., 84 (2010), pp. 733–756.

[9] H. ATTOUCH, G. BUTTAZZO, AND G. MICHAILLE, *Variational Analysis in Sobolev and BV Spaces. Applications to PDEs and Optimization,* MOS-SIAM Ser. Optim. 6, SIAM, Philadelphia, 2006.

[10] M. P. BENDSØE AND O. SIGMUND, *Topology Optimization. Theory, Methods and Applications*, Springer-Verlag, Berlin, 2003.

[11] D. BUCUR AND G. BUTTAZZO, *Variational Methods in Shape Optimization Problems*, Progr. Nonlinear Differential Equations Appl. 65, Birkhäuser Boston, Inc., Boston, MA, 2005.

[12] A. CHERKAEV, *Variational Methods for Structural Optimization*, Appl. Math. Sci. 140, Springer-Verlag, New York, 2000.

[13] S. GARREAU, P. GUILLAUME, AND M. MASMOUDI, *The topological asymptotic for PDE systems: The elasticity case*, SIAM J. Control Optim., 39 (2001), pp. 1756–1778.

[14] B. HACKL, *Methods for reliable topology changes for perimeter-regularized geometric inverse problems*, SIAM J. Numer. Anal., 45 (2007), pp. 2201–2227.

[15] L. HE AND S. OSHER, *Solving the Chan-Vese model by a multiphase level set algorithm based on the topological derivative*, in Proceedings of the 1st International Conference on Scale Space and Variational Methods in Computer Vision, F. Sgallari, A. Murli, and N. Paragios, eds., Lecture Notes in Comput. Sci. 4485, Springer-Verlag, Berlin, 2007, pp. 777–788.

[16] A. HENROT AND M. PIERRE, *Variation et optimisation de formes*, Math. Appl. (Berlin) 48, Springer, Berlin, 2005.

[17] M. HINTERMÜLLER AND A. LAURAIN, *Multiphase image segmentation and modulation recovery based on shape and topological sensitivity*, J. Math. Imaging Vision, 35 (2009), pp. 1–22.

[18] L. HÖRMANDER, *The Analysis of Linear Partial Differential Operators.* I. *Distribution Theory and Fourier Analysis*, Classics in Math., Springer-Verlag, Berlin, 2003.

[19] K. ITO AND K. KUNISCH, *Novel concepts for nonsmooth optimization and their impact on science and technology*, in Proceedings of the International Congress of Mathematicians, Volume IV, Hindustan Book Agency, New Delhi, 2010, pp. 3061–3090.

[20] D. MUMFORD AND J. SHAH, *Optimal approximations by piecewise smooth functions and associated variational problems*, Comm. Pure Appl. Math., 42 (1989), pp. 577–685.

[21] S. A. NAZAROV AND J. SOKOŁOWSKI, *Asymptotic analysis of shape functionals*, J. Math. Pures Appl. (9), 82 (2003), pp. 125–196.

[22] A. A. NOVOTNY, R. A. FEIJÓO, E. TAROCO, AND C. PADRA, *Topological sensitivity analysis*, Comput. Methods Appl. Mech. Engrg., 192 (2003), pp. 803–829.

[23] J. SOKOŁOWSKI AND A. ŻOCHOWSKI, *On the topological derivative in shape optimization*, SIAM J. Control Optim., 37 (1999), pp. 1251–1272.

[24] J. SOKOŁOWSKI AND J.-P. ZOLÉSIO, *Introduction to Shape Optimization. Shape Sensitivity Analysis*, Springer Ser. Comput. Math. 16, Springer-Verlag, Berlin, 1992.