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CRACK DETECTION BY THE TOPOLOGICAL GRADIENT METHOD

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ABSTRACT. The topological sensitivity analysis consists in studying the behavior of a shape functional when modifying the topology of the domain. In general, the perturbation under consideration is the creation of a small hole. In this paper, the topological asymptotic expansion is obtained for the Laplace equation with respect to the insertion of a short crack inside a plane domain. This result is illustrated by some numerical experiments in the context of crack detection.

1. INTRODUCTION

The detection of geometrical faults is a problem of great interest for engineers, to check the integrity of structures for example. The present work deals with the detection and localization of cracks for a simple model problem: the steady-state heat equation (Laplace equation) with the heat flux imposed and the temperature measured on the boundary.

On the theoretical level, the first study on the identifiability of cracks was carried out by A. Friedman and M.S. Vogelius [13]. It was later completed by G. Alessandrini *et al* [2] and A. Ben Abda *et al* [4, 7] who also proved stability results. In the same time, several reconstruction algorithms were proposed [33, 6, 10, 8, 11].

Concurrently, shape optimization techniques have progressed a lot. In particular, some topological optimization methods have been developed for designing domains whose topology is a priori unknown [3, 9, 35]. Among them, the topological gradient method was introduced by A. Schumacher [35] in the context of compliance minimization. Then J. Sokolowski and A. Zochowski [36] generalized it to more general shape functionals by involving an adjoint state. To present the basic idea, let us consider a variable domain Ω of $\mathbb{R}^2$ and a cost functional $j(\Omega) = J(u_\Omega)$ to be minimized, where u_Ω is solution to a given PDE defined over Ω. For a small parameter $\rho \geq 0$, let $\Omega \setminus \overline{B(x_0,\rho)}$ be the perturbed domain obtained by the creation of a circular hole of radius ρ around the point x_0. The topological sensitivity analysis provides an asymptotic expansion of $j(\Omega \setminus \overline{B(x_0,\rho)})$ when ρ tends to zero in the form:

$$j(\Omega \setminus \overline{B(x_0,\rho)}) - j(\Omega) = f(\rho)g(x_0) + o(f(\rho)).$$

In this expression, $f(\rho)$ denotes an explicit positive function going to zero with ρ, $g(x_0)$ is called the topological gradient or topological derivative and it can be computed easily. Consequently, to minimize the criterion j, one has to create holes at some points $\tilde{x}$ where $g(\tilde{x})$ is negative. The topological asymptotic expression has been obtained for various problems, arbitrary shaped holes and a large class of cost functionals. Notably, one can cite the papers [15, 17, 18, 32] where such formulas are proved by using a functional framework based on a domain truncation technique and a generalization of the adjoint method [25].

The theoretical part of this paper deals with the topological sensitivity analysis for the Laplace equation with respect to the insertion of an arbitrary shaped crack with a Neumann condition prescribed on its boundary. In this situation, the contributions focus on the behavior of the solution or of special criterions like the energy integral or the eigenvalues [26, 27, 20]. To calculate the topological derivative, we construct an appropriate adjoint method that applies in the functional space defined over the cracked domain. This approach, combined with a suitable

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approximation of the solution by a double layer potential, leads to a simpler mathematical analysis than the truncation technique. The numerical part is devoted to the inverse geometrical problem described above. The Kohn-Vogelius criterion [21] is used like a cost functional. We explain the procedure as well as presenting some numerical results.

2. Problem formulation

Let Ω be a bounded domain of $\mathbb{R}^2$ with smooth boundary Γ. We consider a regular division $\Gamma = \Gamma_0 \cup \overline{\Gamma_1}$, where Γ_0 and Γ_1 are open manifolds, Γ_0 is of nonzero measure and $\Gamma_0 \cap \Gamma_1 = \emptyset$. The source terms consist in two functions $f \in L^2(\Omega)$ and $g \in H_{00}^{1/2}(\Gamma_1)'$. We recall that, for an open manifold Σ such that $\bar{\Sigma} \subset \tilde{\Sigma}$ where $\tilde{\Sigma}$ is a smooth, open and bounded manifold of the same dimension as Σ, we have [24]

$$H_{00}^{1/2}(\Sigma) = \left\{ u_{|\Sigma}, u \in H^{1/2}(\tilde{\Sigma}), u_{|\tilde{\Sigma}\setminus\bar{\Sigma}} = 0 \right\}. \tag{1}$$

It is endowed with the norm defined for all $u \in H^{1/2}(\tilde{\Sigma})$ by

$$\|u_{|\Sigma}\|_{H_{00}^{1/2}(\Sigma)} = \|u\|_{H^{1/2}(\tilde{\Sigma})}.$$

The initial problem (for the safe domain) is the following: find $u_0 \in H^1(\Omega)$ such that

$$\left\{ \begin{array}{rcll} -\Delta u_0 &=& f & \text{in} \quad \Omega, \\ u_0 &=& 0 & \text{on} \quad \Gamma_0, \\ \partial_n u_0 &=& g & \text{on} \quad \Gamma_1. \end{array} \right. \tag{2}$$

For a given $x_0 \in \Omega$, let us now consider the cracked domain $\Omega_\rho = \Omega \setminus \overline{\sigma_\rho}$, $\sigma_\rho = x_0 + \rho\sigma$, where $\rho > 0$ and σ is a fixed bounded manifold of dimension 1 and of class $\mathcal{C}^1$ (see Figure 1). We assume that Ω_ρ is connected. Possibly changing the coordinate system, we will suppose for convenience that $x_0 = 0$. The new solution $u_\rho \in H^1(\Omega_\rho)$ satisfies

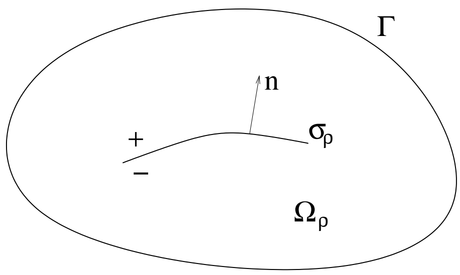

Figure 1. The cracked domain.

$$\left\{ \begin{array}{rcll} -\Delta u_\rho &=& f & \text{in} \quad \Omega_\rho, \\ u_\rho &=& 0 & \text{on} \quad \Gamma_0, \\ \partial_n u_\rho &=& g & \text{on} \quad \Gamma_1, \\ \partial_n u_\rho &=& 0 & \text{on} \quad \sigma_\rho. \end{array} \right. \tag{3}$$

The variational formulation of this problem reads: find $u_\rho \in H^1(\Omega_\rho)$ such that

$$a_\rho(u_\rho, v) = l_\rho(v) \qquad \forall v \in \mathcal{V}_\rho, \tag{4}$$

with

$$\mathcal{V}_\rho = \{u \in H^1(\Omega_\rho), u_{|\Gamma_0} = 0\} \tag{5}$$

and for all $u, v \in \mathcal{V}_\rho$,

$$\begin{cases} a_\rho(u,v) = \displaystyle\int_{\Omega_\rho} \nabla u . \nabla v dx, \\ l_\rho(v) = \displaystyle\int_{\Omega_\rho} f v dx + \int_{\Gamma_1} g v ds. \end{cases} \tag{6}$$

As usual in analysis, the duality product between $H^{1/2}_{00}(\Gamma_1)'$ and $H^{1/2}_{00}(\Gamma_1)$ is denoted by an integral. When $\rho = 0$, that formulation is also valid for Problem (2) by setting $\Omega_0 = \Omega$ in Equations (5) and (6).

Let D be a fixed open set containing the origin and such that $D \subset \Omega$. We define the functional space

$$\mathcal{W} = \{u \in L^2(\Omega), u \in H^1(\Omega \setminus \overline{D})\}, \tag{7}$$

which is equipped with the norm

$$\|u\|_{\mathcal{W}} = (\|u\|^2_{0,\Omega} + \|u\|^2_{1,\Omega\setminus\overline{D}})^{1/2}.$$

In all the paper, for a given domain $\mathcal{O}$, we denote by $\|u\|_{0,\mathcal{O}}$ and $\|u\|_{1,\mathcal{O}}$ the standard norms of the function u in the spaces $L^2(\mathcal{O})$ and $H^1\mathcal{O})$, respectively. The semi-norm $|u|_{1,\mathcal{O}} = \|\nabla u\|_{0,\mathcal{O}}$ will also be used. Consider finally a differentiable functional $J : \mathcal{W} \to \mathbb{R}$. We wish to study the asymptotic behavior when ρ tends to zero of the criterion

$$j(\rho) = J(u_\rho).$$

3. An appropriate adjoint method

The following adjoint method is especially constructed to apply to the above problem. In fact, the key point is that the functional spaces fit together as follows: for all $\rho > 0$,

$$\mathcal{V}_0 \subset \mathcal{V}_\rho \subset \mathcal{W}. \tag{8}$$

For all $\rho \geq 0$, we denote by v_ρ the solution to the problem: find $v_\rho \in \mathcal{V}_\rho$ such that

$$a_\rho(u, v_\rho) = -DJ(u_0)u \qquad \forall u \in \mathcal{V}_\rho. \tag{9}$$

The functions u_0 and v_0 are respectively called the direct and adjoint states. We assume that the following hypothesis holds.

Hypothesis 1. *There exist $\delta \in \mathbb{R}$ and $f : \mathbb{R}_+ \to \mathbb{R}_+$ tending to zero with ρ such that*

(1) $\|u_\rho - u_0\|_{\mathcal{W}} = O(f(\rho))$,
(2) $a_\rho(u_0 - u_\rho, v_\rho) = f(\rho)\delta + o(f(\rho))$.

Then, the asymptotic expansion of $j(\rho)$ is provided by the following Proposition.

Proposition 1. *If Hypothesis 1 is satisfied, then*

$$j(\rho) - j(0) = f(\rho)\delta + o(f(\rho)).$$

Proof. Using the differentiability of J, Hypothesis 1 and Equation (9) , we obtain successively

$$\begin{aligned} j(\rho) - j(0) &= J(u_\rho) - J(u_0) \\ &= DJ(u_0)(u_\rho - u_0) + o(\|u_\rho - u_0\|_{\mathcal{W}}) \\ &= -a_\rho(u_\rho - u_0, v_\rho) + o(f(\rho)) \\ &= f(\rho)\delta + o(f(\rho)). \end{aligned}$$

□

4. Asymptotic calculus

We have now to check Hypothesis 1 in the context of Problem (3). To simplify the presentation, all technical estimates are reported in Section 5. In this way, we assume for the moment that $\|u_\rho - u_0\|_{\mathcal{W}} = O(\rho^2)$, which insures that the first condition of Hypothesis 1 is fulfilled if $\rho^2 = O(f(\rho))$. We focus here on the determination of $f(\rho)$ and δ such that the second part of Hypothesis 1 holds.

4.1. **Preliminary calculus.** We obtain by using the Green formula

$$\begin{aligned} a_\rho(u_0 - u_\rho, v_\rho) &= \int_{\Omega_\rho} \nabla(u_0 - u_\rho).\nabla v_\rho dx \\ &= -\int_{\sigma_\rho} \partial_n u_0 [v_\rho] ds \end{aligned}$$

where $[v_\rho] = v_{\rho|\sigma_\rho^+} - v_{\rho|\sigma_\rho^-} \in H^{1/2}_{00}(\sigma_\rho)$ (see Figure 1). We introduce the variation

$$w_\rho = v_\rho - v_0.$$

From (9), we obtain that w_ρ is solution to the problem : find $w_\rho \in H^1(\Omega_\rho)$ such that

$$\left\{ \begin{array}{rcllr} \Delta w_\rho &=& 0 & \text{in} & \Omega_\rho, \\ w_\rho &=& 0 & \text{on} & \Gamma_0, \\ \partial_n w_\rho &=& 0 & \text{on} & \Gamma_1, \\ \partial_n w_\rho &=& -\partial_n v_0 & \text{on} & \sigma_\rho. \end{array} \right. \tag{10}$$

We are going to search for an appropriate approximation of w_ρ.

4.2. **Definitions and standard results about exterior problems.** Let Σ be a bounded manifold of dimension 1, of class $\mathcal{C}^1$ and $\Lambda = \mathbb{R}^2 \setminus \overline{\Sigma}$. We suppose that Λ is connected. The space $W^1(\Lambda)$ is defined by (see *e.g.* [19, 23, 16]):

$$W^1(\Lambda) = \left\{ u \in \mathcal{D}'(\Lambda), \frac{u}{(1+r)\ln(1+r)} \in L^2(\Lambda) \text{ and } \nabla u \in L^2(\Lambda) \right\}.$$

It is equipped with the norm

$$\|u\|_{W^1(\Lambda)} = \left(\left\| \frac{u}{(1+r)\ln(1+r)} \right\|^2_{L^2(\Lambda)} + \|\nabla u\|^2_{L^2(\Lambda)} \right)^{1/2}.$$

In the above expressions, the letter r denotes the distance to the origin.

Given $\psi \in H^{1/2}_{00}(\Sigma)'$, let us now consider the problem

$$\left\{ \begin{array}{rcllr} \Delta u &=& 0 & \text{in} & \Lambda, \\ u &=& 0 & \text{at} & \infty, \\ \partial_n u &=& \psi & \text{on} & \Sigma. \end{array} \right. \tag{11}$$

To solve it with the help of a potential, we need to introduce the fundamental solution of the Laplacian in 2D:

$$E(x) = \frac{1}{2\pi} \ln |x|.$$

We have the following theorem (see [16, 30]).

Theorem 2. (1) *Problem (11) has a unique solution $u \in W^1(\Lambda)$ and the map $\psi \mapsto u$ is linear and continuous from $H^{1/2}_{00}(\Sigma)'$ into $W^1(\Lambda)$.*

(2) *The solution u is the double layer potential*

$$u(x) = \int_\Sigma \eta(y) \partial_{n_y} E(x-y) ds(y) \qquad \forall x \in \Lambda,$$

where $\eta = T_\Sigma \psi$, T_Σ being a known isomorphism from $H^{1/2}_{00}(\Sigma)'$ into $H^{1/2}_{00}(\Sigma)$.

(3) *We have the jump relation for the same orientation as in Figure 1:*

$$[u] = u_{|\Sigma^+} - u_{|\Sigma^-} = -\eta.$$

(4) *If Σ is a line segment with curvilinear abscissa s, we have for all $\eta \in (H_{00}^{1/2} \cap \mathcal{C}^1)(\Sigma)$ and $\varphi \in \mathcal{D}(\Sigma)$*

$$< T_\Sigma^{-1}\eta, \varphi >= -\int_\Sigma \int_\Sigma \frac{d\eta}{ds}(x)\frac{d\varphi}{ds}(y)E(x-y)ds(x)ds(y).$$

4.3. **Estimate of w_ρ.** Let us now come back to the approximation of the solution to Problem (10).

(1) *First approximation :* We approximate w_ρ by h_ρ the solution to the exterior problem: find $h_\rho \in W^1(\mathbb{R}^2 \setminus \overline{\sigma_\rho})$ such that

$$\left\{\begin{array}{rcll} \Delta h_\rho &=& 0 & \text{in} \quad \mathbb{R}^2 \setminus \overline{\sigma_\rho}, \\ \partial_n h_\rho &=& -\partial_n v_0 & \text{on} \quad \sigma_\rho, \\ h_\rho &=& 0 & \text{at} \quad \infty. \end{array}\right. \tag{12}$$

Then, we use the change of variable

$$h_\rho(x) = \rho H_\rho\left(\frac{x}{\rho}\right).$$

The function $H_\rho \in W^1(\mathbb{R}^2 \setminus \overline{\sigma})$ verifies

$$\left\{\begin{array}{rcll} \Delta H_\rho &=& 0 & \text{in} \quad \mathbb{R}^2 \setminus \overline{\sigma}, \\ \partial_n H_\rho(x) &=& -\partial_n v_0(\rho x) & \text{on} \quad \sigma, \\ H_\rho &=& 0 & \text{at} \quad \infty. \end{array}\right.$$

By Theorem 2, H_ρ can be written in the form

$$H_\rho(x) = \int_\sigma q_\rho(y)\partial_{n_y}E(x-y)ds(y) \qquad \forall x \in \mathbb{R}^2 \setminus \overline{\sigma}, \tag{13}$$

where $q_\rho \in H_{00}^{1/2}(\sigma)$ is defined by

$$q_\rho = T_\sigma(-\partial_n v_0(\rho x)). \tag{14}$$

(2) *Second approximation :* We approximate now q_ρ by

$$q = T_\sigma(-\nabla v_0(0).\mathbf{n}). \tag{15}$$

4.4. **Asymptotic expansion of the cost functional.** We set

$$E_1(\rho) = -\int_{\sigma_\rho} \partial_n u_0[w_\rho - h_\rho]ds.$$

Then

$$\begin{array}{rcl} a_\rho(u_0 - u_\rho, v_\rho) &=& -\displaystyle\int_{\sigma_\rho} \partial_n u_0[w_\rho]ds \\ &=& -\displaystyle\int_{\sigma_\rho} \partial_n u_0[h_\rho]ds + E_1(\rho) \\ &=& -\rho^2 \displaystyle\int_\sigma \partial_n u_0(\rho x)[H_\rho]ds + E_1(\rho). \end{array}$$

We denote also

$$E_2(\rho) = -\rho^2 \int_\sigma \partial_n u_0(\rho x)(q_\rho - q)ds.$$

By the jump relation of Theorem 2, we have

$$\begin{aligned} a_\rho(u_0 - u_\rho, v_\rho) &= \rho^2 \int_\sigma \partial_n u_0(\rho x) q_\rho ds + E_1(\rho) \\ &= \rho^2 \int_\sigma \partial_n u_0(\rho x) q ds + E_1(\rho) + E_2(\rho). \end{aligned}$$

Finally, we define

$$E_3(\rho) = \rho^2 \int_\sigma (\partial_n u_0(\rho x) - \nabla u_0(0).\mathbf{n}) q ds$$

and we obtain

$$a_\rho(u_0 - u_\rho, v_\rho) = \rho^2 \int_\sigma \nabla u_0(0).\mathbf{n} q ds + E_1(\rho) + E_2(\rho) + E_3(\rho).$$

We will prove in Section 5 that $E_i(\rho) = o(\rho^2)$ $\forall i = 1, 2, 3$. Therefore, we are allowed to set

$$f(\rho) = \rho^2,$$

$$\delta = \nabla u_0(0). \int_\sigma q \mathbf{n} ds.$$

Let us introduce the so-called polarization matrix A_σ, defined as the matrix of the linear map

$$V \in \mathbb{R}^2 \mapsto A_\sigma V = \int_\sigma T_\sigma(V.\mathbf{n}) \mathbf{n} ds. \tag{16}$$

In the case of a hole instead of a crack, similar matrices can be defined with the help of a single layer potential [34, 31, 14, 5, 29]. They are proved to be symmetric positive definite, and this is still true for a crack. Then, we can write

$$\delta = -\nabla u_0(0).A_\sigma \nabla v_0(0).$$

From Proposition 1, we derive the following theorem.

Theorem 3. *If*

- *the cost functional J is differentiable on the space $\mathcal{W}$ defined by (7),*
- *the source terms f and $DJ(u_0)$ are of regularity H^2 in a neighborhood of the origin,*
- *the direct and adjoint states are solutions to (4) and (9) with a_ρ and l_ρ defined by (6),*
- *the polarization matrix A_σ is defined by (16),*

then the criterion admits the following asymptotic expansion when ρ tends to zero:

$$j(\rho) - j(0) = -\rho^2 \nabla u_0(0).A_\sigma \nabla v_0(0) + o(\rho^2). \tag{17}$$

4.5. **Straight crack.** Let σ be a line segment of length 2 centered at the origin, with unit normal $\mathbf{n}$. Using Theorem 2, one can check that the appropriate density evaluated at the curvilinear abscissa s is

$$T_\sigma(V.\mathbf{n})(s) = 2(V.\mathbf{n})\sqrt{1 - s^2}.$$

We have then

$$A_\sigma V = \pi(V.\mathbf{n})\mathbf{n}.$$

Corollary 4. *For a straight crack of normal $\mathbf{n}$, the topological asymptotic expansion reads*

$$j(\rho) - j(0) = -\pi\rho^2 (\nabla u_0(0).\mathbf{n})(\nabla v_0(0).\mathbf{n}) + o(\rho^2). \tag{18}$$

This formula extends to the case of a vector field. Denoting by u_0^i and v_0^i, $i = 1..P$ the components of u_0 and v_0, one gets the expansion:

$$j(\rho) - j(0) = -\pi\rho^2 \sum_{i=1}^{P} (\nabla u_0^i(0).\mathbf{n})(\nabla v_0^i(0).\mathbf{n}) + o(\rho^2). \tag{19}$$

5. Proofs

5.1. Preliminary lemmas.

Lemma 1. *Consider $\psi \in H_{00}^{1/2}(\sigma)'$ and let $z \in W^1(\mathbb{R}^2 \setminus \overline{\sigma})$ be the solution to the problem*

$$\left\{\begin{array}{rcll} \Delta z &=& 0 & \text{in} \quad \mathbb{R}^2 \setminus \overline{\sigma}, \\ z &=& 0 & \text{at} \quad \infty, \\ \partial_n z &=& \psi & \text{on} \quad \sigma. \end{array}\right.$$

there exists $c > 0$, independent of ρ and ψ, such that

$$|z|_{1,\frac{1}{\rho}(\Omega\setminus\overline{D})} \leq c\rho \|\psi\|_{H_{00}^{1/2}(\sigma)'}.$$

Proof. According to Theorem 2, there exists $\eta \in H_{00}^{1/2}(\sigma)$ such that

$$z(x) = \int_\sigma \eta(y)\partial_{n_y} E(x-y)ds(y), \ \forall x \in \mathbb{R}^2 \setminus \overline{\sigma},$$

where $\eta = T_\sigma \psi$. Using a Taylor expansion of E computed at the point x and the continuity of T_σ, we have that

$$|\nabla z(x)| \leq \frac{c}{|x|^2} \|\psi\|_{H_{00}^{1/2}(\sigma)'},$$

from which we deduce the result. □

Lemma 2. *Consider $g \in H_{00}^{1/2}(\Gamma_1)'$, $\rho \geq 0$, $h \in H_{00}^{1/2}(\sigma_\rho)'$ and let $z \in H^1(\Omega_\rho)$ be the solution to the problem*

$$\left\{\begin{array}{rcl} \Delta z &=& 0 \ \text{in} \ \Omega_\rho, \\ z &=& 0 \ \text{on} \ \Gamma_0, \\ \partial_n z &=& g \ \text{on} \ \Gamma_1, \\ \partial_n z &=& h \ \text{on} \ \sigma_\rho. \end{array}\right. \tag{20}$$

There exist some positive constants denoted by c independent of ρ, g and h such that for all ρ small enough

$$\|z\|_{0,\Omega_\rho} \leq c\rho^2 \|h(\rho x)\|_{H_{00}^{1/2}(\sigma)'} + c\|g\|_{H_{00}^{1/2}(\Gamma_1)'},$$
$$|z|_{1,\Omega_\rho} \leq c\rho \|h(\rho x)\|_{H_{00}^{1/2}(\sigma)'} + c\|g\|_{H_{00}^{1/2}(\Gamma_1)'},$$
$$\|z\|_{1,\Omega\setminus\overline{D}} \leq c\rho^2 \|h(\rho x)\|_{H_{00}^{1/2}(\sigma)'} + c\|g\|_{H_{00}^{1/2}(\Gamma_1)'}.$$

Proof. The function z is split into $z_1 + z_2$ respectively solutions to

$$\left\{\begin{array}{rcll} \Delta z_1 &=& 0 & \text{in} \quad \mathbb{R}^2 \setminus \overline{\sigma_\rho}, \\ z_1 &=& 0 & \text{at} \quad \infty, \\ \partial_n z_1 &=& h & \text{on} \quad \sigma_\rho, \end{array}\right. \qquad \left\{\begin{array}{rcll} \Delta z_2 &=& 0 & \text{in} \quad \Omega_\rho, \\ z_2 &=& -z_1 & \text{on} \quad \Gamma_0, \\ \partial_n z_2 &=& g - \partial_n z_1 & \text{on} \quad \Gamma_1, \\ \partial_n z_2 &=& 0 & \text{on} \quad \sigma_\rho. \end{array}\right.$$

The function $\tilde{z}_1(x) = z_1(\rho x)/\rho$ is solution to

$$\left\{\begin{array}{rcll} \Delta \tilde{z}_1 &=& 0 & \text{in} \quad \mathbb{R}^2 \setminus \overline{\sigma}, \\ \tilde{z}_1 &=& 0 & \text{at} \quad \infty, \\ \partial_n \tilde{z}_1 &=& h(\rho x) & \text{on} \quad \sigma. \end{array}\right.$$

By elliptic regularity, we have

$$\|\tilde{z}_1\|_{W^1(\mathbb{R}^2\setminus\overline{\sigma})} \leq c\|h(\rho x)\|_{H_{00}^{1/2}(\sigma)'}.$$

Lemma 1 yields

$$|\tilde{z}_1|_{1,\frac{1}{\rho}(\Omega\setminus\overline{D})} \leq c\rho \|h(\rho x)\|_{H_{00}^{1/2}(\sigma)'}.$$

Then, a change of variable brings

$$\|z_1\|_{0,\Omega_\rho} \leq c\rho^2 \|h(\rho x)\|_{H_{00}^{1/2}(\sigma)'},$$

$$|z_1|_{1,\Omega_\rho} \leq c\rho \|h(\rho x)\|_{H_{00}^{1/2}(\sigma)'},$$
$$\|z_1\|_{1,\Omega\setminus\overline{D}} \leq c\rho^2 \|h(\rho x)\|_{H_{00}^{1/2}(\sigma)'}.$$

Moreover, we have by elliptic regularity

$$\|z_2\|_{1,\Omega_\rho} \leq c\|z_1\|_{1,\Omega\setminus\overline{D}} + c\|g\|_{H_{00}^{1/2}(\Gamma_1)'},$$

which completes the proof. □

5.2. **Proof of Theorem 3.** The result is a consequence of Proposition 1 if we prove that $\|u_\rho - u_0\|_{\mathcal{W}} = O(\rho^2)$ and that $E_i(\rho) = o(\rho^2)$ for $i = 1, 2, 3$.

5.2.1. *Estimate of the variation of the solution.* It is an immediate application of Lemma 2 that

$$\|u_\rho - u_0\|_{\mathcal{W}} = O(\rho^2).$$

5.2.2. *Estimate of the remainders.* We will denote by c any positive constant independent of ρ.

(1) We have

$$\begin{aligned} |E_1(\rho)| &= \rho \left| \int_\sigma \partial_n u_0(\rho x)[(w_\rho - h_\rho)(\rho x)] ds \right| \\ &\leq \rho \|\partial_n u_0(\rho x)\|_{H_{00}^{1/2}(\sigma)'} \|[(w_\rho - h_\rho)(\rho x)]\|_{H_{00}^{1/2}(\sigma)} \\ &\leq c\rho \|[e_\rho(\rho x)]\|_{H_{00}^{1/2}(\sigma)}, \end{aligned}$$

where $e_\rho = w_\rho - h_\rho$ is solution to

$$\left\{ \begin{array}{rcll} \Delta e_\rho &=& 0 & \text{in } \Omega_\rho, \\ e_\rho &=& -h_\rho & \text{on } \Gamma_0, \\ \partial_n e_\rho &=& -\partial_n h_\rho & \text{on } \Gamma_1, \\ \partial_n e_\rho &=& 0 & \text{on } \sigma_\rho. \end{array} \right.$$

Denoting by B some ball containing σ, we obtain by using the trace theorem

$$\|[e_\rho(\rho x)]\|_{H_{00}^{1/2}(\sigma)} = \inf_{\gamma\in\mathbb{R}} \|[e_\rho(\rho x) + \gamma]\|_{H_{00}^{1/2}(\sigma)} \leq c \inf_{\gamma\in\mathbb{R}} \|e_\rho(\rho x) + \gamma\|_{1,B\setminus\overline{\sigma}} \leq c|e_\rho(\rho x)|_{1,B\setminus\overline{\sigma}}.$$

A change of variable and the elliptic regularity yield

$$\|[e_\rho(\rho x)]\|_{H_{00}^{1/2}(\sigma)} \leq c|e_\rho|_{1,\Omega_\rho} \leq c \inf_{\gamma\in\mathbb{R}} \|e_\rho + \gamma\|_{1,\Omega_\rho} \leq c \inf_{\gamma\in\mathbb{R}} \|h_\rho + \gamma\|_{1,\Omega\setminus\overline{D}} \leq c|h_\rho|_{1,\Omega\setminus\overline{D}}.$$

Next, a change of variable and Lemma 1 yield

$$\|[e_\rho(\rho x)]\|_{H_{00}^{1/2}(\sigma)} \leq c\rho |H_\rho|_{1,\frac{1}{\rho}(\Omega\setminus\overline{D})} \leq c\rho^2 \|\partial_n v_0(\rho x)\|_{H_{00}^{1/2}(\sigma)'}.$$

Finally,

$$|E_1(\rho)| \leq c\rho^3.$$

(2) We have

$$\begin{aligned} |E_2(\rho)| &\leq \rho^2 \|\partial_n u_0(\rho x)\|_{H_{00}^{1/2}(\sigma)'} \|q_\rho - q\|_{H_{00}^{1/2}(\sigma)} \\ &\leq c\rho^2 \|q_\rho - q\|_{H_{00}^{1/2}(\sigma)}. \end{aligned}$$

By continuity of the operator T_σ, we have

$$\begin{aligned} \|q_\rho - q\|_{H_{00}^{1/2}(\sigma)} &\leq c\|\partial_n v_0(\rho x) - \nabla v_0(0).\mathbf{n}\|_{H_{00}^{1/2}(\sigma)'} \\ &\leq c\|\partial_n v_0(\rho x) - \nabla v_0(0).\mathbf{n}\|_{\mathcal{C}^0(\sigma)}. \end{aligned}$$

Yet, v_0 is of class $\mathcal{C}^2$ in a neighborhood of the origin. Thus,

$$\|q_\rho - q\|_{H_{00}^{1/2}(\sigma)} \leq c\rho \tag{21}$$

and

$$|E_2(\rho)| \leq c\rho^3.$$

(3) We have

$$|E_3(\rho)| \leq \rho^2 \|\partial_n u_0(\rho x) - \nabla u_0(0).\mathbf{n}\|_{H^{1/2}_{00}(\sigma)'} \|q\|_{H^{1/2}_{00}(\sigma)}.$$

As u_0 is of class $\mathcal{C}^2$ in a neighborhood of the origin,

$$\|\partial_n u_0(\rho x) - \nabla u_0(0).\mathbf{n}\|_{H^{1/2}_{00}(\sigma)'} \leq \|\partial_n u_0(\rho x) - \nabla u_0(0).\mathbf{n}\|_{\mathcal{C}^0(\sigma)} \leq c\rho.$$

Hence,

$$|E_3(\rho)| \leq c\rho^3.$$

□

6. Numerical applications

In this numerical study, we use Formula (19) to detect and locate cracks with the help of boundary measurements. The context is the one of the steady-state heat equation.

6.1. **The inverse problem.** Let Ω be a domain containing a perfectly insulating crack σ^* whose location, orientation, shape and length are to be retrieved. We dispose of the temperature θ measured on the boundary Γ for a heat flux φ prescribed: $\theta = u(\sigma^*)_{|\Gamma}$, where $u(\sigma^*)$ is the solution to the PDE

$$\left\{ \begin{array}{rcll} \Delta u(\sigma^*) & = & 0 & \text{in} \quad \Omega \setminus \overline{\sigma}, \\ \partial_n u(\sigma^*) & = & \varphi & \text{on} \quad \Gamma, \\ \partial_n u(\sigma^*) & = & 0 & \text{on} \quad \sigma. \end{array} \right. \tag{22}$$

To ensure well-posedness of the above system, we assume the normalization condition

$$\int_\Gamma \varphi ds = 0$$

and we impose that the mean value of the solution is equal to zero:

$$\int_{\Omega \setminus \overline{\sigma^*}} u(\sigma^*) dx = 0.$$

In practice, several measurements, corresponding to different fluxes, may be needed. But for the clarity of the presentation, let us consider the simplest case of one measurement.

6.2. **The cost functional and the topological gradient.** Since the boundary conditions (θ, φ) are overspecified, one can define for any crack $\sigma \subset \Omega$ two forward problems:

- the "Dirichlet" problem:

$$\left\{ \begin{array}{rcll} \Delta u_D(\sigma) & = & 0 & \text{in} \quad \Omega \setminus \overline{\sigma}, \\ u_D(\sigma) & = & \theta & \text{on} \quad \Gamma, \\ \partial_n u_D(\sigma) & = & 0 & \text{on} \quad \sigma, \end{array} \right. \tag{23}$$

- the "Neumann" problem:

$$\left\{ \begin{array}{rcll} \Delta u_N(\sigma) & = & 0 & \text{in} \quad \Omega \setminus \overline{\sigma}, \\ \partial_n u_N(\sigma) & = & \varphi & \text{on} \quad \Gamma, \\ \partial_n u_N(\sigma) & = & 0 & \text{on} \quad \sigma. \end{array} \right. \tag{24}$$

The solution to this latter system is defined up to an additive constant, which is determined by the equation

$$\int_{\Omega \setminus \overline{\sigma}} u_N(\sigma) dx = 0. \tag{25}$$

This condition plays the same role as the fact of prescribing a Dirichlet condition on a part of the boundary, which was chosen for simplicity in the theoretical study. The actual crack

σ^* is reached ($\sigma = \sigma^*$) when there is no misfit between both solutions, that is, when the cost functional

$$\mathcal{J}(\sigma) = J(u_D(\sigma), u_N(\sigma)) = \frac{1}{2}\|u_D(\sigma) - u_N(\sigma)\|^2_{L^2(\Omega)} \tag{26}$$

vanishes. This is the so-called Kohn-Vogelius criterion [21]. To compute the corresponding topological gradient, we need to solve numerically:

- the two direct problems on the safe domain

$$\begin{cases} \Delta u_D &= 0 & \text{in} & \Omega, \\ u_D &= \theta & \text{on} & \Gamma, \end{cases} \tag{27}$$

$$\begin{cases} \Delta u_N &= 0 & \text{in} & \Omega, \\ \partial_n u_N &= \varphi & \text{on} & \Gamma, \\ \displaystyle\int_\Omega u_N dx = 0, & & & \end{cases} \tag{28}$$

whose solutions are denoted by u_D and u_N instead of $u_D(\emptyset)$ and $u_N(\emptyset)$ to simplify the writing,

- two adjoint problems (defined on the safe domain too)

$$\begin{cases} -\Delta v_D &= -(u_D - u_N) & \text{in} & \Omega, \\ v_D &= 0 & \text{on} & \Gamma, \end{cases} \tag{29}$$

$$\begin{cases} -\Delta v_N &= +(u_D - u_N) - \overline{u_D} & \text{in} & \Omega, \\ \partial_n v_N &= 0 & \text{on} & \Gamma, \\ \displaystyle\int_\Omega v_N dx = 0, & & & \end{cases} \tag{30}$$

with

$$\overline{u_D} = \frac{1}{\text{meas }(\Omega)} \int_\Omega u_D dx.$$

The above adjoint problems are derived directly from their variational formulations (a quotient functional space is needed to define the Neumann problem). The existence of the solution to Problem (30) comes from Equation (25). Using a vector field $U = (u_D, u_N)$, Corollary 4 provides the following expression of the topological asymptotic for that cost functional and the insertion of a small straight crack:

$$\mathcal{J}(\sigma_{x,\rho,\mathbf{n}}) - \mathcal{J}(\emptyset) = -\pi\rho^2[(\nabla u_D(x).\mathbf{n})(\nabla v_D(x).\mathbf{n}) + (\nabla u_N(x).\mathbf{n})(\nabla v_N(x).\mathbf{n})] + o(\rho^2),$$

where $\sigma_{x,\rho,\mathbf{n}}$ is the line crack of length 2ρ, centered at the point x and of unit normal $\mathbf{n}$. One can also write the corresponding topological gradient

$$g(x, \mathbf{n}) = -\pi[(\nabla u_D(x).\mathbf{n})(\nabla v_D(x).\mathbf{n}) + (\nabla u_N(x).\mathbf{n})(\nabla v_N(x).\mathbf{n})]$$

as follows:

$$g(x, \mathbf{n}) = \mathbf{n}^T M(x)\mathbf{n},$$

where $M(x)$ is the symmetric matrix defined by

$$M(x) = -\pi\text{sym }(\nabla u_D(x) \otimes \nabla v_D(x) + \nabla u_N(x) \otimes \nabla v_N(x)).$$

The notation sym (X) stands for the symmetric part of the square matrix X: sym $(X) = (X + X^T)/2$ and the tensor product of two vectors means $U \otimes V = UV^T$. According to that expression, $g(x, \mathbf{n})$ is minimal at the point x when the normal $\mathbf{n} = \mathbf{n_1}$ is an eigenvector associated to the smallest eigenvalue $\lambda_1(x)$ of the matrix $M(x)$. Then, $g(x, \mathbf{n_1}) = \lambda_1(x)$. Henceforth, we will call topological gradient this value.

6.3. **Numerical result in one iteration without noise.** Let us now describe a simple and very fast numerical procedure. First, we solve the two direct problems and the two adjoint problems (Dirichlet and Neumann). Then, in each cell of the mesh, we compute the matrix $M(x)$ and its eigenvalues. By regarding the unknown crack as the addition of small straight cracks whose interactions are neglected and by using the previous asymptotic analysis, one expects that crack to lie in the regions where the topological gradient is the most negative.

Let Ω be the unit disc and σ^* be a line segment crack. The heat flux φ is imposed on Γ by $\varphi(x) = x_2$, the second coordinate of the point x. In this experiment, the flux inside the safe domain is not parallel to the crack, so that only one measurement is needed for the reconstruction (see [4]). We apply the procedure described above. The location of the unknown crack as well as the topological gradient are indicated in Figures 2 and 3. We observe that the most negative values of the topological gradient are located near the actual crack.

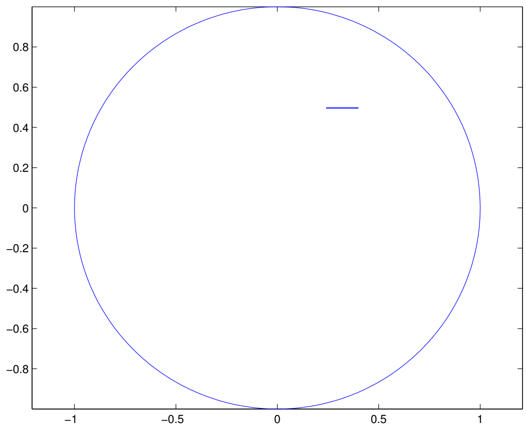

FIGURE 2. The unknown crack.

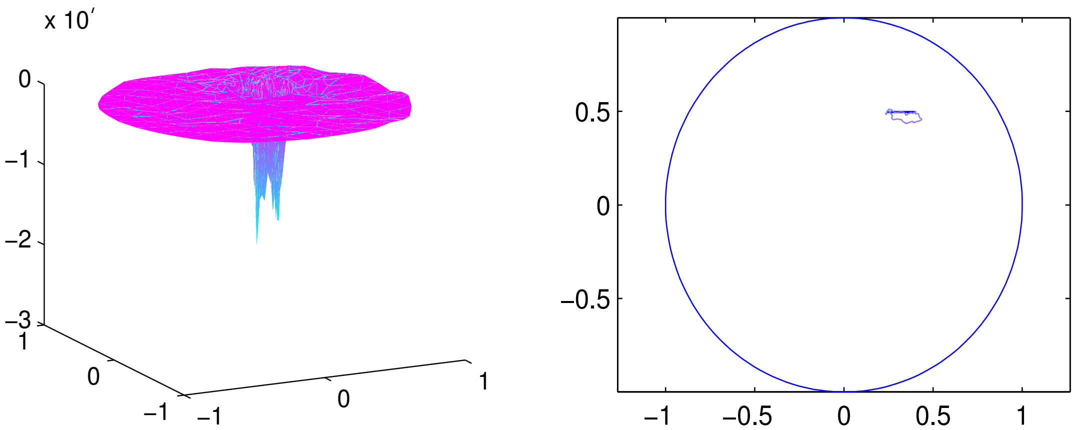

FIGURE 3. On the left: the topological gradient ; on the right: superposition of the actual crack and a negative isovalue of the topological gradient.

6.4. **Numerical results in one iteration with noise.**

6.4.1. *Case of a single crack.* We focus here on simulated noisy measurements. A white noise is added to the exact data. Figure 4 shows the results obtained for 5%, 10% and 20% of noise. We observe that the inversion procedure is quite robust with respect to the presence of noise in the measurements.

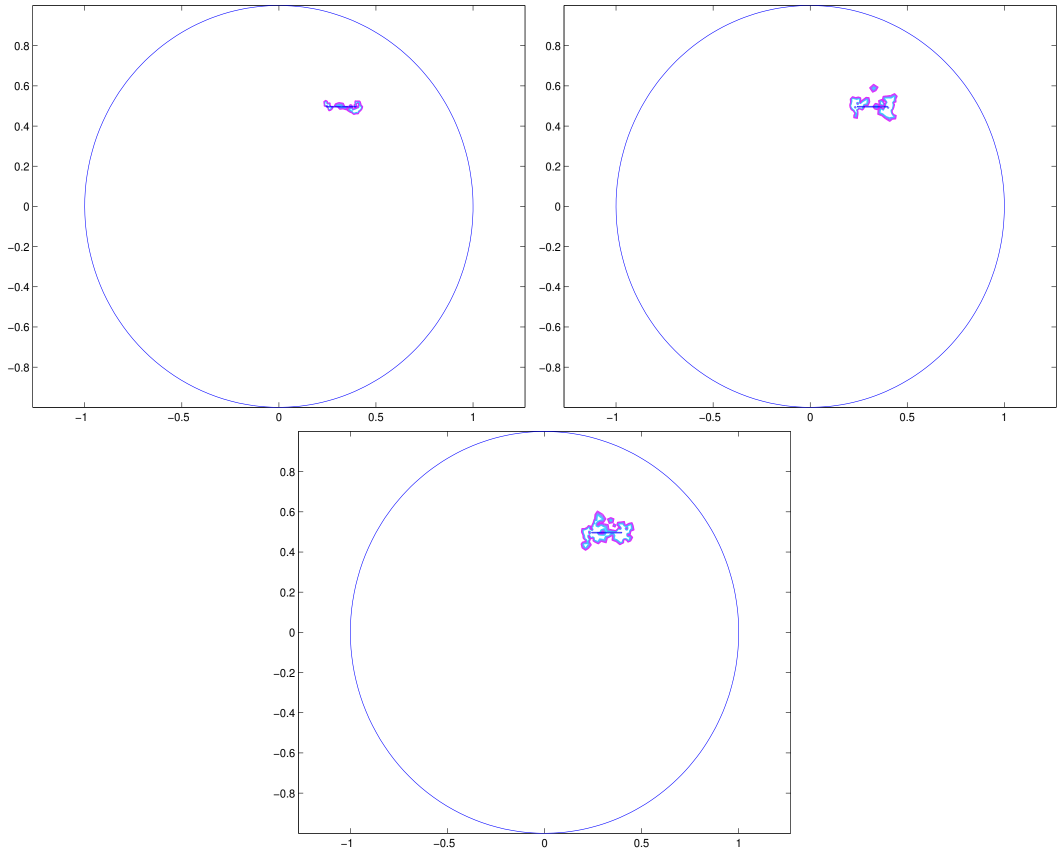

FIGURE 4. Representation of a negative isovalue of the topological gradient for 5%, 10% and 20% of noise, respectively.

6.4.2. *Case of multi-cracks.* The computation of the topological gradient does not depend on the number of cracks inside the domain. This remark is illustrated by the following experiment. The actual cracks and the topological gradient are represented in Figure 5. We use now two fluxes $\varphi_1(x) = x_1$ and $\varphi_2(x) = x_2$. We take as a cost functional the sum of the two quadratic misfits. Hence, the matrix $M(x)$ is assembled by adding the two corresponding contributions. We emphasize that these results are again obtained in only one iteration.

6.5. **Identification of cracks with incomplete data.** It is a more realistic situation where a part only of the border is accessible to measurements. Let Ω be the unit disc with boundary $\Gamma = \Gamma_0 \cup \Gamma_1$. The heat flux φ is prescribed on Γ and the temperature θ is measured on Γ_1, here a quarter of the whole boundary. For any crack $\sigma \subset \Omega$, we consider the two following problems:

- the "Neumann-Dirichlet" problem:

$$\left\{ \begin{array}{rcccl} \Delta u_D(\sigma) & = & 0 & \text{in} & \Omega \setminus \overline{\sigma}, \\ u_D(\sigma) & = & \theta & \text{on} & \Gamma_1, \\ \partial_n u_D(\sigma) & = & \varphi & \text{on} & \Gamma_0, \\ \partial_n u_D(\sigma) & = & 0 & \text{on} & \sigma, \end{array} \right. \tag{31}$$

- the "Neumann" problem:

$$\left\{ \begin{array}{rcccl} \Delta u_N(\sigma) & = & 0 & \text{in} & \Omega \setminus \overline{\sigma}, \\ \partial_n u_N(\sigma) & = & \varphi & \text{on} & \Gamma, \\ \partial_n u_N(\sigma) & = & 0 & \text{on} & \sigma, \end{array} \right. \tag{32}$$

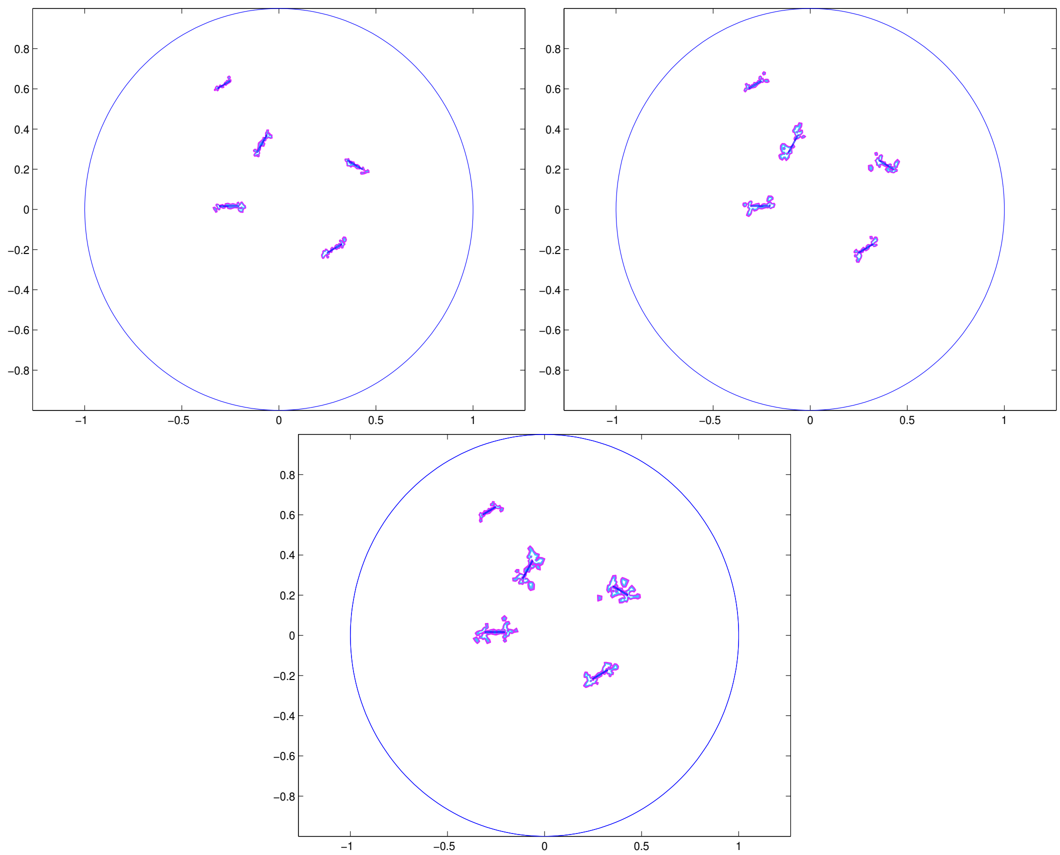

FIGURE 5. Respectively 5%, 10% and 20% of noise.

with the normalization condition

$$\int_{\Omega\setminus\overline{\sigma}} u_N(\sigma)dx = \int_{\Omega\setminus\overline{\sigma}} u_D(\sigma)dx.$$

We use the same cost functional as before (see Equation (26)), but for the above fields. Hence we have the same topological gradient expression and the numerical procedure remains unchanged. The results are represented in Figure 6. The cracks are located in a satisfactory manner.

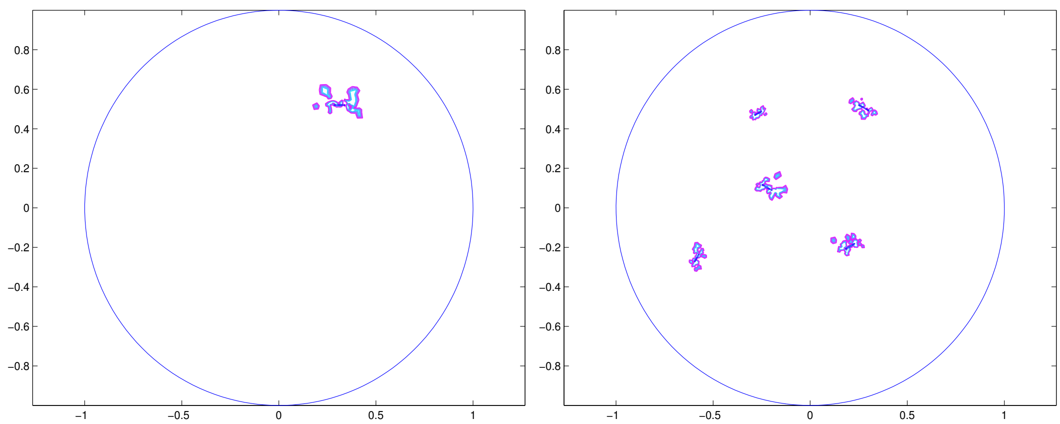

FIGURE 6. Topological gradient with incomplete data (no noise).

6.6. **An iterative method.** The algorithm consists in inserting at each iteration an insulating element (that is, numerically, an element whose thermal conductivity is very small) where the topological gradient is the most negative. The process is stopped when the cost functional does not decrease any more.

Algorithm
Initialization: Choose the initial domain Ω_0 and create a mesh which will remain fixed during the process. That domain is identified with the set of its finite elements: $\Omega_0 = \{x_n, n = 1, ..., N\}$. Set $k = 0$.
Repeat:

(1) Solve the direct and adjoint problems in Ω_k,
(2) Compute the topological gradient g_k,
(3) Search for the minimum of the topological gradient: $y_k = \text{argmin}(g_k(x), x \in \Omega_k)$,
(4) Set $\Omega_{k+1} = \Omega_k \setminus \{y_k\}$,
(5) $k \leftarrow k + 1$.

We wish here to recover two cracks with the help of one flux $\varphi(x) = x_2$ (complete data, no additive noise). The final image and the convergence history of the cost functional are represented in Figure 7.

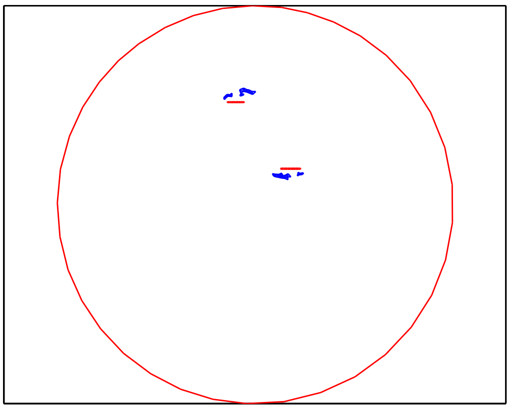
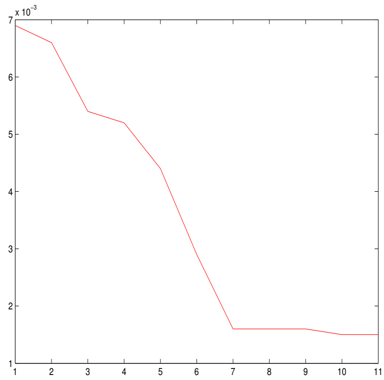

Figure 7. On the left: the actual cracks and the reconstructed cracks after a few iterations ; on the right: the convergence history of the criterion.

7. Conclusion

The mathematical framework presented in this paper can be adapted to determine the sensitivity with respect to the insertion of a small crack for a large class of linear and elliptic problems.

The topological gradient leads to fast methods for detecting and locating cracks in that it only requires to solve the direct and adjoint problems and satisfactory results are obtained after a small number of iterations performed on a fixed grid. These methods can provide a good initial guess for more accurate classical shape optimization algorithms [22, 33].

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