TOPOLOGICAL SENSITIVITY ANALYSIS FOR SOME NONLINEAR PDE SYSTEMS

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ABSTRACT. The aim of the topological sensitivity analysis is to determine an asymptotic expansion of a design functional when creating a small hole inside the domain. In this work, such an expansion is obtained for a certain class of nonlinear PDE systems of order 2 in dimensions 2 and 3 with a Dirichlet condition prescribed on the boundary of an arbitrarily shaped hole. Some examples of such operators are presented.
KEY WORDS. shape optimization, topological sensitivity, nonlinear PDE.

RÉSUMÉ. L'analyse de sensibilité topologique consiste à rechercher un développement asymptotique d'une fonctionnelle de forme par rapport à la création d'un petit trou dans le domaine. Dans ce travail, on établit un tel développement pour une certaine famille d'EDP non linires d'ordre 2 en dimensions 2 et 3 et une condition de Dirichlet imposée au bord d'un trou de forme quelconque. Des exemples d'opérateurs de ce type sont présentés.
MOTS-CLÉS. optimisation de forme, sensibilité topologique, EDP non linéaires.

1. INTRODUCTION

The topological sensitivity analysis aims to provide an asymptotic expansion of a shape functional with respect to the size of a small hole created inside the domain. For a criterion $j(\Omega) = J_\Omega(u_\Omega)$ where $\Omega \subset \mathbb{R}^N$ ($N = 2$ or 3) and u_Ω is the solution of a set of partial differential equations defined over Ω, this expansion can be generally written in the form

$$j(\Omega \setminus (\overline{x_0 + \rho\omega})) - j(\Omega) = f(\rho)g(x_0) + o(f(\rho)). \tag{1}$$

In this expression, ρ and x_0 denote respectively the radius and the center of the hole, ω is a fixed domain containing the origin and $f(\rho)$ is a positive function going to zero with ρ. The function g is commonly called "topological gradient", or "topological derivative".

The first asymptotic analyses of solutions of boundary value problems defined in singularly perturbed domains go back to the works of Il'in [9] and Nazarov [16] who introduced the methods of matched and compound asymptotic expansions, respectively. Since that times, these methods have been developed towards rather complicated situations (see the books [10, 14]), even including some nonlinear problems (see also the original paper [13]) and have been applied to the asymptotic study of special objective functions, namely the energy integral and the eigenvalues of the operator. This concept of topological sensitivity of a shape functional was introduced in the field of shape optimization by Schumacher [24] who calculated the topological derivative of the compliance in linear elasticity and used it for locating the best places to remove matter in the structure. Then several methods have been worked out to derive the topological asymptotic expansion (1) for various problems and general cost functions. The most significant are briefly recalled below.

The first one was proposed by Sokolowski and Zochowski [25], and further developed by Novotny *et al.* [20]. The principle is to start from the variation of the shape functional corresponding to an infinitesimal growth of an existing hole, which is given by the classical shape optimization theory [15, 26], and then to pass to the limit when the initial hole vanishes. The main difficulty lies in the determination of a sufficiently accurate approximation of the spatial derivatives of the solution on the border of the hole, which are involved in the shape derivative.

Another approach, instigated by Masmoudi [12], consists in reformulating the problem in a fixed domain by means of a truncation technique. Then, a generalization of the adjoint method is used to evaluate the variation of the criterion. This framework enabled to derive the topological asymptotic expansions for several problems: linear elasticity [5], Poisson [6], Stokes [7], quasi-Stokes [8] and Helmholtz [23, 22] equations. The last contributions we shall cite in this list are the papers of Nazarov and Sokolowski [17, 18, 19], placed in the context of a PDE with linear and homogeneous differential operator. The first one deals with more general shape functionals in 3D by means of an appropriate approximation of the solution in the sense of weighted Hölder norms. The two others concern the peculiar case of a Dirichlet condition on the hole in 2D for which the function $f(\rho) = |\ln \rho|^{-1}$ goes very slowly to zero. The authors obtained higher order terms by using a tricky extension of the operator concentrating the perturbation at a point (see also [10, 14] for the application of the methods of matched and compound asymptotic expansions at an arbitrary order). However, the related numerical procedure remains to be developed.

The present paper addresses the case of a state equation associated to a differential operator of the form

$$P(u) = -\tilde{\Delta}u + \Phi(u),$$

where u is a vector field, $\tilde{\Delta}$ is a linear and homogeneous differential operator of order 2 and Φ is a possibly nonlinear function mapping an element of $H^1(\Omega)$ to an element of $H^1(\Omega)'$ and satisfying additional technical assumptions. This class includes notably the linear operators cited before, a nonlinear Helmholtz equation and the Navier-Stokes equations for incompressible fluids. For this latter case, the reader is referred to [2] for a complete proof. An homogeneous Dirichlet condition is prescribed on the hole. In order to avoid a truncation which would raise technical difficulties because of the nonlinearity, the PDE is reformulated in the whole domain with the solution extended by zero inside the hole. This leads to a singularly perturbed variational problem requiring a further generalization of the adjoint method. In 3D, the solution is approximated following the methodology of compound asymptotic expansions combined with the solution of the exterior limit problem with the help of a single layer potential. This approximation is valid in the sense of the Sobolev norms, which are sufficient to treat the most standard shape functionals. It is proved that the topological gradient depends only on the principal part $\tilde{\Delta}$ of the operator and on the shape of the hole through a polarization tensor. The dimension $N = 2$ is the so-called critical dimension (according to the terminology of [14]) because of the logarithmic behavior of the fundamental solution. The dominant part of the solution is not driven by boundary layers, which results in the fact that the topological gradient is independant of the shape of the hole.

The paper is organized as follows. The problem of interest is formulated in Section 2. For simplicity, the scalar case with $\tilde{\Delta} = \Delta$ is first considered, then generalized. The adjoint method is described in Section 3. The asymptotic analysis and the main results are presented in Section 4. Some examples of shape functionals are exhibited. For the sake of readability, all technical proofs are reported in Section 5. Section 6 is devoted to the application of the previous results to the nonlinear Helmholtz equation.

2. Problem presentation

2.1. **The initial boundary value problem.** Let Ω be an open, bounded and connected subset of $\mathbb{R}^N$, $N = 2$ or 3, with smooth boundary Γ and consider a function Φ that maps an element of $H^1(\mathcal{O})$ to an element of $H^1(\mathcal{O})'$ for any open and bounded subset $\mathcal{O}$ of $\mathbb{R}^N$. To insure that this function is well-defined, we assume that the following property holds: if $\mathcal{O} = \mathcal{O}_1 \cup \mathcal{O}_2 \cup (\partial\mathcal{O}_1 \cap \partial\mathcal{O}_2)$, $\mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset$, then for all $u, v \in H^1(\mathcal{O})$ we have

$$< \Phi(u), v >_{H^1(\mathcal{O})', H^1(\mathcal{O})} = < \Phi(u_{|\mathcal{O}_1}), v_{|\mathcal{O}_1} >_{H^1(\mathcal{O}_1)', H^1(\mathcal{O}_1)} + < \Phi(u_{|\mathcal{O}_2}), v_{|\mathcal{O}_2} >_{H^1(\mathcal{O}_2)', H^1(\mathcal{O}_2)} \cdot \tag{2}$$

Given $\sigma \in L^2(\Omega)$, we consider a scalar field $u_0 \in H_0^1(\Omega)$ which is assumed to be the unique solution of the PDE

$$\begin{cases} -\Delta u_0 + \Phi(u_0) & = & \sigma & \text{in} & \Omega, \\ u_0 & = & 0 & \text{on} & \Gamma. \end{cases} \tag{3}$$

For reasons that will appear later, we suppose that σ is continuous in Ω, that $u_0 \in H_{loc}^2(\Omega)$ and that the map $u \in H_0^1(\Omega) \mapsto \Phi(u) \in H^{-1}(\Omega)$ is Fréchet-differentiable at the point u_0. The variational formulation of the above problem reads

$$\begin{cases} u_0 \in H_0^1(\Omega), \\ F_0(u_0) & = & 0, \end{cases} \tag{4}$$

where F_0 is the map defined by

$$\begin{cases} F_0 : H_0^1(\Omega) & \longrightarrow & H^{-1}(\Omega), \\ < F_0(u), v > & = & \displaystyle\int_\Omega \nabla u.\nabla v dx + < \Phi(u) - \sigma, v > & \forall u, v \in H_0^1(\Omega). \end{cases} \tag{5}$$

Here and in all the sequel, the brackets $< .,. >$ denote the duality product between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

2.2. **The perturbed boundary value problem.** Let ω be an open and bounded subset of $\mathbb{R}^N$ containing the origin, with smooth and connected boundary $\partial\omega$, and let $x_0 \in \Omega$. For any sufficiently small parameter $\rho > 0$, consider the perforated domain $\Omega_\rho = \Omega \setminus \overline{\omega_\rho}$ where $\omega_\rho = x_0 + \rho\omega$. Possibly shifting the origin of the coordinate system, we assume for convenience that $x_0 = 0$. The perturbed field $u_\rho \in H_0^1(\Omega_\rho)$ is supposed to be of regularity H^2 in the vicinity of ω_ρ and to be the unique solution of the system

$$\begin{cases} -\Delta u_\rho + \Phi(u_\rho) & = & \sigma & \text{in} & \Omega_\rho, \\ u_\rho & = & 0 & \text{on} & \Gamma, \\ u_\rho & = & 0 & \text{on} & \partial\omega_\rho. \end{cases} \tag{6}$$

We define now the map

$$\begin{cases} F_\rho : H_0^1(\Omega) & \longrightarrow & H^{-1}(\Omega), \\ < F_\rho(u), v > & = & < F_0(u), v > + \displaystyle\int_{\partial\omega_\rho} \partial_n u_\rho v ds + \int_{\omega_\rho} \sigma v dx & \forall u, v \in H_0^1(\Omega). \end{cases} \tag{7}$$

Due to the Green formula, u_ρ satisfies

$$\begin{cases} u_\rho \in H_0^1(\Omega), & u_{\rho|\omega_\rho} = 0, \\ F_\rho(u_\rho) & = & 0. \end{cases} \tag{8}$$

In this system, u_ρ stands actually for the extension by zero inside ω_ρ of the function u_ρ defined previously. The same notation has been kept to simplify the writing. That convention, consisting in considering as canonical the imbedding $H_0^1(\Omega_\rho) \hookrightarrow H_0^1(\Omega)$, will be implicitly used throughout all the paper.

2.3. **The topological sensitivity problem.** We consider a cost functional $j(\rho) = J_\rho(u_\rho)$ where J_ρ is a differentiable map from $H_0^1(\Omega)$ into $\mathbb{R}$. We wish to study the asymptotic behavior of the variation $j(\rho) - j(0)$ when ρ tends to zero. To do so, we start by introducing an appropriate adjoint method.

Remark 1. *The Dirichlet condition on Γ could be replaced without any influence on the topological sensitivity analysis by any boundary condition such that Problems (3) and (6) remain well-posed in the sense of existence, uniqueness and elliptic regularity. This can be seen in the proofs, which require only a continuous dependence of the solutions with respect to the data.*

3. An appropriate adjoint method

The asymptotic expansion of the cost functional will be provided by the following theorem, presented here in an abstract setting with suitable hypotheses. The checking of these assumptions for the problem presented above will be carried out in Sections 4 and 5.

Theorem 1. *Let $\mathcal{V}$ be a Hilbert space on the real field. For all $\rho \in \mathbb{R}_+$, we consider*

- *a differentiable map $F_\rho : \mathcal{V} \to \mathcal{V}'$,*
- *an element $u_\rho \in \mathcal{V}$ satisfying*

$$F_\rho(u_\rho) = 0, \tag{9}$$

- *a differentiable functional $J_\rho : \mathcal{V} \to \mathbb{R}$.*

We assume that there exists $v_0 \in \mathcal{V}$, called adjoint state, solving

$$< DF_0(u_0)\varphi, v_0 >_{\mathcal{V}',\mathcal{V}} = -DJ_0(u_0)\varphi \qquad \forall \varphi \in \mathcal{V}. \tag{10}$$

We suppose moreover that there exist four real numbers δ_{F1}, δ_{F2}, δ_{J1} and δ_{J2}, as well as a function $f(\rho)$ tending to zero with ρ such that, when $\rho \to 0$,

$$< F_\rho(u_\rho) - F_0(u_\rho), v_0 > = f(\rho)\delta_{F1} + o(f(\rho)), \tag{11}$$

$$< F_0(u_\rho) - F_0(u_0) - DF_0(u_0)(u_\rho - u_0), v_0 > = f(\rho)\delta_{F2} + o(f(\rho)), \tag{12}$$

$$J_\rho(u_\rho) - J_0(u_\rho) = f(\rho)\delta_{J1} + o(f(\rho)), \tag{13}$$

$$J_0(u_\rho) - J_0(u_0) - DJ_0(u_0)(u_\rho - u_0) = f(\rho)\delta_{J2} + o(f(\rho)). \tag{14}$$

Then we have the asymptotic expansion

$$J_\rho(u_\rho) - J_0(u_0) = f(\rho)(\delta_{F1} + \delta_{F2} + \delta_{J1} + \delta_{J2}) + o(f(\rho)).$$

Proof. Thanks to Equation (9) we can write

$$J_\rho(u_\rho) - J_0(u_0) = J_\rho(u_\rho) - J_0(u_0) + < F_\rho(u_\rho) - F_0(u_0), v_0 > .$$

Next, Equations (11), (12), (13) and (14) yield

$$\begin{aligned} J_\rho(u_\rho) - J_0(u_0) &= DJ_0(u_0)(u_\rho - u_0) + f(\rho)(\delta_{J1} + \delta_{J2}) + o(f(\rho)) \\ &+ < DF_0(u_0)(u_\rho - u_0), v_0 > + f(\rho)(\delta_{F1} + \delta_{F2}) + o(f(\rho)). \end{aligned}$$

Using Equation (10) we obtain the announced result. $\qquad \square$

4. Main results

Our purpose now is to show that Theorem 1 applies to derive the topological sensitivity expression for the problem described in Section 2. We present here the different steps of the asymptotic analysis, leading to the main results of the paper gathered in Theorems 2, 3 and 4. To simplify the presentation, all technical proofs are reported in Section 5.

Theorem 1 will be applied to the family of maps $(F_\rho)_{\rho \geq 0}$ defined by (5) and (7). The functional space involved is $\mathcal{V} = H_0^1(\Omega)$. For convenience, we introduce the notation

$$R_u(v) = \Phi(u + v) - \Phi(u). \tag{15}$$

Let us first consider the three-dimensional case ($N = 3$).

4.1. **Topological sensitivity in 3D.** The first step of the asymptotic analysis consists in determining an appropriate approximation of the variation $u_\rho - u_0$.

4.1.1. *Asymptotic behavior of the solution.*

(1) *Approximation by the solution of an exterior problem.* We split the variation of the solution into

$$u_\rho - u_0 = h_\rho + r_\rho, \tag{16}$$

where h_ρ and r_ρ solve

$$\begin{cases} -\Delta h_\rho = 0 & \text{in} \quad \mathbb{R}^3 \setminus \overline{\omega_\rho}, \\ h_\rho \to 0 & \text{at} \quad \infty, \\ h_\rho = -u_0 & \text{on} \quad \partial\omega_\rho, \end{cases}$$

and

$$\begin{cases} -\Delta r_\rho + R_{u_0+h_\rho}(r_\rho) = -R_{u_0}(h_\rho) & \text{in} \quad \Omega_\rho, \\ r_\rho = -h_\rho & \text{on} \quad \Gamma, \\ r_\rho = 0 & \text{on} \quad \partial\omega_\rho. \end{cases}$$

The dominant part of $u_\rho - u_0$ for the needed norms is expected to be provided by h_ρ. The remainder r_ρ will be estimated later. Next, we set $H_\rho(x) = h_\rho(\rho x)$, which solves

$$\begin{cases} -\Delta H_\rho = 0 & \text{in} \quad \mathbb{R}^3 \setminus \overline{\omega}, \\ H_\rho \to 0 & \text{at} \quad \infty, \\ H_\rho = -u_0(\rho x) & \text{on} \quad \partial\omega. \end{cases}$$

(2) *Approximation of the boundary condition on the hole.* We split H_ρ into $H_\rho = H + S_\rho$ (H is expected to be the leading term) with

$$\begin{cases} -\Delta H = 0 & \text{in} \quad \mathbb{R}^3 \setminus \overline{\omega}, \\ H \to 0 & \text{at} \quad \infty, \\ H = -u_0(0) & \text{on} \quad \partial\omega, \end{cases}$$

and

$$\begin{cases} -\Delta S_\rho = 0 & \text{in} \quad \mathbb{R}^3 \setminus \overline{\omega}, \\ S_\rho \to 0 & \text{at} \quad \infty, \\ S_\rho = -u_0(\rho x) + u_0(0) & \text{on} \quad \partial\omega. \end{cases}$$

That function H can be explicited with the help of a single layer potential [3]:

$$H(x) = \int_{\partial\omega} E(x-y)\eta(y)ds(y) \qquad \forall x \in \mathbb{R}^3 \setminus \overline{\omega},$$

where the fundamental solution of the Laplace operator in 3D is

$$E(x) = \frac{1}{4\pi|x|}$$

and the density $\eta \in H^{-1/2}(\partial\omega)$ is the unique solution of the boundary integral equation

$$\int_{\partial\omega} E(x-y)\eta(y)ds(y) = -u_0(0) \qquad \forall x \in \partial\omega. \tag{17}$$

4.1.2. *Asymptotic behavior of the cost functional.* We have to determine $f(\rho)$, δ_{F1}, δ_{F2}, δ_{J1} and δ_{J2} such that Equations (11)-(14) hold true. The values of δ_{J1} and δ_{J2} are given in Section 4.6 for some examples of cost functional. We assume for the moment that $\delta_{F2} = 0$ and we focus on the calculus of δ_{F1}. According to Equation (7), we have

$$< F_\rho(u_\rho) - F_0(u_\rho), v_0 > = \int_{\partial\omega_\rho} \partial_n u_\rho v_0 ds + \int_{\omega_\rho} \sigma v_0 dx.$$

Replacing u_ρ by $u_0 + h_\rho + r_\rho$, making a change of variable and replacing successively H_ρ by $H + S_\rho$ and $v_0(\rho x)$ by $v_0(0) + [v_0(\rho x) - v_0(0)]$, we obtain

$$< F_\rho(u_\rho) - F_0(u_\rho), v_0 > = \rho \left(\int_{\partial\omega} \partial_n H ds \right) v_0(0) + \sum_{i=1}^{4} \mathcal{E}_i(\rho)$$

with

$$\mathcal{E}_1(\rho) = \int_{\partial\omega_\rho} \partial_n u_0 v_0 ds + \int_{\omega_\rho} \sigma v_0 dx,$$

$$\mathcal{E}_2(\rho) = \int_{\partial\omega_\rho} \partial_n r_\rho v_0 ds,$$

$$\mathcal{E}_3(\rho) = \rho \int_{\partial\omega} \partial_n S_\rho v_0(\rho x) ds,$$

$$\mathcal{E}_4(\rho) = \rho \int_{\partial\omega} \partial_n H[v_0(\rho x) - v_0(0)] ds.$$

Next, due to the jump relation of the single layer potential, we have $\partial_n H = -\eta$ on $\partial\omega$. Assuming that $|\mathcal{E}_i(\rho)| = o(\rho)$ for $i = 1, ..., 4$, which will be proved in Section 5, we deduce that Equation (11) holds with

$$f(\rho) = \rho \qquad \text{and} \qquad \delta_{F1} = -\left(\int_{\partial\omega} \eta ds\right) v_0(0).$$

Thanks to the linearity of Equation (17), the expression of δ_{F1} can be rewritten with the help of the coefficient

$$\mathcal{P}_\omega = \int_{\partial\omega} \hat{\eta} ds, \tag{18}$$

where $\hat{\eta} \in H^{-1/2}(\partial\omega)$ is the unique solution of the integral equation

$$\int_{\partial\omega} E(x - y)\hat{\eta}(y) ds(y) = 1 \qquad \forall x \in \partial\omega. \tag{19}$$

Thus, under the following hypothesis needed to estimate the errors $\mathcal{E}_i(\rho)$ and to prove that $\delta_{F2} = 0$, the asymptotic expansion of the cost functional can be derived from Theorem 1, constituting Theorem 2.

Hypothesis 1.   (1) *There exists $\lambda > 0$ and some constant $c > 0$ such that for any $f \in H^{-1}(\Omega_\rho)$, $\varphi \in H^{1/2}(\Gamma)$ and $u \in H^1(\Omega_\rho)$ with $\|u\|_{1,\Omega_\rho} < \lambda$ and $\|f\|_{-1,\Omega_\rho}$, $\|\varphi\|_{1/2,\Gamma}$ small enough, the problem*

$$\begin{cases} -\Delta v + R_u(v) &= f & in & \Omega_\rho, \\ v &= \varphi & on & \Gamma, \\ v &= 0 & on & \partial\omega_\rho, \end{cases} \tag{20}$$

admits one and only one solution satisfying

$$\|v\|_{1,\Omega_\rho} \le c(\|f\|_{-1,\Omega_\rho} + \|\varphi\|_{1/2,\Gamma}).$$

(2) *There exists some constant $c' > 0$ such that for all $v \in H^1(\Omega)$ with $\|v\|_{1,\Omega}$ small enough,*

$$\|R_{u_0}(v)\|_{-1,\Omega} \le c'(\|v\|_{0,\Omega} + \|v\|_{1,\Omega}^2).$$

Here and in the sequel, the direct state u_0 is considered as fixed. Thus c' may depend on u_0.

(3) *If u is of class $\mathcal{C}^2$, then $\Phi(u)$ is of class $\mathcal{C}^0$.*

(4) *When $\|v\|_{1,\Omega}$ tends to zero, $v \in H_0^1(\Omega)$, we have*

$$< R_{u_0}(v) - DR_{u_0}(0)v, v_0 > = o(\|v\|_{0,\Omega} + \|v\|_{1,\Omega}^2).$$

Some examples of such functions Φ are given in Section 4.4. As counter-examples, there are the differential operators of order 2 which are not defined from $H^1(\mathcal{O})$ to $H^1(\mathcal{O})'$ for any open and bounded set $\mathcal{O}$.

To write the classical formulation of Problem (10), we need to introduce the adjoint operator $D\Phi(u_0)^*$ of the differential $D\Phi(u_0) : H_0^1(\Omega) \to H^{-1}(\Omega)$, which is defined by

$$< D\Phi^*(u_0)\psi, \varphi > = < D\Phi(u_0)\varphi, \psi > \quad \forall \varphi, \psi \in H_0^1(\Omega).$$

Theorem 2 (Topological sensitivity in 3D). *If*

- *the function Φ satisfies Hypothesis 1 and $\|u_0\|_{1,\Omega} < \lambda$,*
- *the cost functional satisfies Equations (13) and (14) with $f(\rho) = \rho$,*
- *the adjoint problem: find $v_0 \in H_0^1(\Omega)$ such that*

$$\begin{cases} -\Delta v_0 + D\Phi(u_0)^* v_0 & = & -DJ_0(u_0) & in & \Omega, \\ v_0 & = & 0 & on & \Gamma, \end{cases} \tag{21}$$

   has at least one solution,
- *the direct and adjoint states u_0 and v_0 are of class $\mathcal{C}^2$ in the vicinity of the origin,*
- *the coefficient $\mathcal{P}_\omega$ is defined by (18),*

then the following asymptotic expansion holds true:

$$j(\rho) - j(0) = \rho \left[\mathcal{P}_\omega u_0(0) v_0(0) + \delta_{J1} + \delta_{J2} \right] + o(\rho). \tag{22}$$

4.2. Topological sensitivity in 2D.

4.2.1. *Asymptotic behavior of the solution.* In dimension 2, the fundamental solution of the Laplacian reads

$$E(x) = \frac{-1}{2\pi} \ln|x|.$$

It does not tend to zero at infinity. Hence, an approximation of $u_\rho - u_0$ by a single layer potential is not relevant. We adopt a very different approach.

We split $u_\rho - u_0$ into

$$u_\rho - u_0 = h_\rho + r_\rho + s_\rho \tag{23}$$

where the expected dominant part near the hole is

$$h_\rho(x) = -\frac{E(x)}{E(\rho)} u_0(0)$$

and r_ρ, s_ρ verify

$$\begin{cases} -\Delta r_\rho + R_{u_0 + h_\rho}(r_\rho) & = & -R_{u_0}(h_\rho) & in & \Omega, \\ r_\rho & = & -h_\rho & on & \Gamma, \end{cases}$$

$$\begin{cases} -\Delta s_\rho + R_{u_0 + h_\rho + r_\rho}(s_\rho) & = & 0 & in & \Omega_\rho, \\ s_\rho & = & 0 & on & \Gamma, \\ s_\rho & = & -u_0 - h_\rho - r_\rho & on & \partial\omega_\rho. \end{cases}$$

In a natural way, we have denoted for simplicity $E(\rho) = -\ln\rho/2\pi$.

4.2.2. *Asymptotic behavior of the cost functional.* Denoting by

$$\mathcal{E}_1(\rho) = \int_{\partial\omega_\rho} \partial_n u_0 v_0 ds + \int_{\omega_\rho} \sigma v_0 dx,$$

$$\mathcal{E}_2(\rho) = \int_{\partial\omega_\rho} \partial_n r_\rho v_0 ds,$$

$$\mathcal{E}_3(\rho) = \int_{\partial\omega_\rho} \partial_n s_\rho v_0 ds,$$

$$\mathcal{E}_4(\rho) = \int_{\partial\omega_\rho} \partial_n h_\rho [v_0 - v_0(0)] ds,$$

we obtain

$$
\begin{aligned}
< F_\rho(u_\rho) - F_0(u_\rho), v_0 > \ &= \ \left(\int_{\partial\omega_\rho} \partial_n h_\rho ds \right) v_0(0) + \sum_{i=1}^{4} \mathcal{E}_i(\rho) \\
&= \ -\frac{1}{E(\rho)} \left(\int_{\partial\omega_\rho} \partial_n E ds \right) u_0(0) v_0(0) + \sum_{i=1}^{4} \mathcal{E}_i(\rho) \\
&= \ \frac{u_0(0)v_0(0)}{E(\rho)} + \sum_{i=1}^{4} \mathcal{E}_i(\rho).
\end{aligned}
$$

This latter equality comes straightforwardly from the fact that E is the fundamental solution of the Laplacian. We will prove in Section 5 that $|\mathcal{E}_i(\rho)| = o(-1/\ln\rho)$ for $i = 1, ..., 4$. Thus, we set

$$
f(\rho) = \frac{-1}{\ln\rho} \qquad \text{and} \qquad \delta_{F1} = 2\pi u_0(0) v_0(0).
$$

Here again we have that $\delta_{F2} = 0$. We derive the topological asymptotic expansion from Theorem 1. For the proof, the following hypothesis is required.

Hypothesis 2. (1) *There exists $p \in]1, 2[$ and $q \in]1, +\infty[$ such that Φ can be extended in a map, still denoted by Φ, which is defined from $W^{1,p}(\mathcal{O})$ into $L^q(\mathcal{O})$ for any open and bounded subset $\mathcal{O}$ of $\mathbb{R}^2$.*

(2) *There exists $\lambda > 0$ and some constant $c > 0$ such that for any $f \in H^{-1}(\Omega_\rho)$, $\varphi \in H^{1/2}(\Gamma)$ and $u \in W^{1,p}(\Omega_\rho)$ with $\|u\|_{W^{1,p}(\Omega_\rho)} < \lambda$ and $\|f\|_{-1,\Omega_\rho}$, $\|\varphi\|_{1/2,\Gamma}$ small enough, the problem*

$$
\begin{cases}
-\Delta v + R_u(v) &= f & in & \Omega_\rho, \\
v &= \varphi & on & \Gamma, \\
v &= 0 & on & \partial\omega_\rho,
\end{cases}
\tag{24}
$$

has one and only one solution satisfying

$$
\|v\|_{1,\Omega_\rho} \ \le \ c(\|f\|_{-1,\Omega_\rho} + \|\varphi\|_{1/2,\Gamma}).
$$

(3) *There exists some constant $c' > 0$ such that for any open set $\mathcal{O} \subset \Omega$ and for all $u, v \in W^{1,p}(\mathcal{O})$ with $\|u\|_{W^{1,p}(\mathcal{O})} \le \lambda$ and $\|v\|_{W^{1,p}(\mathcal{O})}$ small enough,*

$$
\|R_u(v)\|_{L^q(\mathcal{O})} \ \le \ c'\|v\|_{W^{1,p}(\mathcal{O})}.
$$

(4) *When $\|v\|_{1,\Omega}$ tends to zero, $v \in H_0^1(\Omega)$, we have*

$$
< R_{u_0}(v) - DR_{u_0}(0)v, v_0 > \ = \ o(\|v\|_{W^{1,p}(\Omega)}).
$$

Theorem 3 (Topological sensitivity in 2D). *If*
- *the function Φ satisfies Hypothesis 2 and $\|u_0\|_{W^{1,p}(\Omega)} < \lambda$,*
- *the cost functional satisfies Equations (13) and (14) with $f(\rho) = -1/\ln\rho$,*
- *the adjoint problem (21) has at least one solution $v_0 \in H_0^1(\Omega)$,*
- *the direct and adjoint states u_0 and v_0 are of class $\mathcal{C}^2$ in the vicinity of the origin,*

then the following asymptotic expansion holds true:

$$
j(\rho) - j(0) \ = \ \frac{-1}{\ln\rho} \left[2\pi u_0(0) v_0(0) + \delta_{J1} + \delta_{J2} \right] + o(\frac{-1}{\ln\rho}).
\tag{25}
$$

4.3. Generalization. The results of Theorems 2 and 3 can be easily generalized to the case where the Laplacian is replaced by an operator $\tilde{\Delta}$ satisfying the following properties.

Hypothesis 3. *For any open and bounded set $\mathcal{O} \subset \mathbb{R}^N$, $\tilde{\Delta}$ is defined by*

$$
\tilde{\Delta} : \begin{array}{ccc}
\mathcal{V}(\mathcal{O}) & \to & \mathcal{V}_0(\mathcal{O})' \\
u & \mapsto & div(A\nabla u),
\end{array}
$$

where

- $\mathcal{V}(\mathcal{O})$ *is a closed subspace of* $H^1(\mathcal{O})^n$, $n \geq 1$,
- $\mathcal{V}_0(\mathcal{O}) = \mathcal{V}(\mathcal{O}) \cap H^1_0(\mathcal{O})^n$,
- *A is a tensor of order 4 such that*

$$AX : X \geq cX : X, \qquad \forall X \in \mathcal{M}_{N,n}(\mathbb{R}),$$

- *the fundamental matrix of* $\tilde{\Delta}$ *satisfies, in the sense of the uniform norm with respect to the angular coordinate* θ,

$$E(x) = O\left(\frac{1}{|x|}\right), \qquad (|x| \to \infty) \qquad in \ 3D,$$

$$E(x) \sim -m_2 \frac{\ln|x|}{2\pi}I, \qquad (|x| \to 0) \qquad in \ 2D,$$

where $m_2 \in \mathbb{R}^*$ *and I is the identity matrix of size* n.

For such a vector operator, the scalar $\mathcal{P}_\omega$ has to be replaced by the $n \times n$ matrix of the linear map

$$X \in \mathbb{R}^n \mapsto \mathcal{P}_\omega X = \int_{\partial\omega} \eta ds,$$

the density η being the unique solution of the boundary integral equation

$$\int_{\partial\omega} E(x-y)\eta(y)ds(y) \;=\; X \qquad \forall x \in \partial\omega. \tag{26}$$

Such a matrix is generally called a polarization matrix [21]. In the present case of a Dirichlet contition on the border of the inclusion, it coincides with the well-known notion in harmonic analysis of capacity matrix.

Then, under the hypotheses of Theorems 2 and 3 satisfied by replacing Δ by $\tilde{\Delta}$, $H^1(\mathcal{O})$ by $\mathcal{V}(\mathcal{O})$ and $H^1_0(\mathcal{O})$ by $\mathcal{V}_0(\mathcal{O})$, we have the topological asymptotic expansions:

$$j(\rho) - j(0) \;\sim\; \rho\left[\mathcal{P}_\omega u_0(0).v_0(0) + \delta_{J1} + \delta_{J2}\right] \qquad in \ 3D, \tag{27}$$

$$j(\rho) - j(0) \;\sim\; \frac{-1}{\ln\rho}\left[\frac{2\pi}{m_2}u_0(0).v_0(0) + \delta_{J1} + \delta_{J2}\right] \qquad in \ 2D. \tag{28}$$

If the solution u_ρ is complex-valued, which means that the Hilbert space $\mathcal{V}(\mathcal{O})$ is complex, then the above results can be adapted by identifying $\mathbb{C}$ with $\mathbb{R}^2$. This leads to the following changes. We refer the reader *e.g.* to [23] for more details.

- The cost functional J_0, which remains real-valued, is supposed to be "$\mathbb{R}$-differentiable" at u_0, *i.e.* there exists $L_0 \in \mathcal{V}_0(\Omega)'$ such that

$$J_0(u_0 + h) - J_0(u_0) = \Re L_0(h) + o(\|h\|).$$

- The adjoint problem reads: find $v_0 \in H^1_0(\Omega)$ such that

$$\begin{cases} -\Delta v_0 + D\Phi(u_0)^* v_0 &=& -L_0 & in & \Omega, \\ v_0 &=& 0 & on & \Gamma. \end{cases} \tag{29}$$

- The dot product $u_0(0).v_0(0)$ in the asymptotic formulas (27) and (28) has to be understood as the real part of the hermitian dot product of $\mathbb{C}^n$.

4.4. **Examples.** We give in Table 1 some examples of differential operators $\tilde{\Delta}$ satisfying Hypothesis 3 and of functions Φ verifying Hypothesis 1 in dimension 3 and Hypothesis 2 in dimension 2. For the linear functions, the checking of these Hypotheses is immediate. For the Navier-Stokes equations (incompressible case), it is quite technical and involves specific proporties of the problem. The reader is referred to [2]. The nonlinear Helmholtz equation is treated in Section 6. The exponent s denotes an integer such that $s \in \{1, 2\}$ in 3D, $s \geq 1$ in 2D. The coefficient ϵ can be any complex number, but we must keep in mind that the PDE (the initial one and also the one in the presence of a small hole) must be well-posed. This can be obtained by a fixed point argument (see *e.g.* [11] for a similar case) provided that the right hand side satisfies $\|\sigma\|_{0,\Omega} \leq c_0 |\epsilon|^{-1/2s}$, c_0 depending on Ω and s. Thus, if $|\epsilon|$ is large, then $\|\sigma\|_{0,\Omega}$ has to be small.

We give in Table 2 the corresponding coefficient m_2. In elasticity, the plane strain case is presented. For plane stress, $\lambda^* = 2\mu\lambda/(\lambda + 2\mu)$ must be substituted for λ. We use the standard notations $r = |x|$ and $\mathbf{e_r} = x/|x|$.

PDE system	$\tilde{\Delta}u$	$\mathcal{V}(\mathcal{O})$	$\Phi(u)$
Laplace/ (nonlinear) Helmholtz	Δu	$H^1(\mathcal{O})^n$	$-k^2(1 + \epsilon|u|^{2s})u$
linear elasticity/elastic waves	div $\sigma(u)$	$H^1(\mathcal{O})^N$	$-k^2 u$
Stokes/quasi-Stokes, Navier-Stokes	$\nu\Delta u$	$\{u \in H^1(\mathcal{O})^N, \text{div } u = 0\}$	$\alpha u, \nabla u.u$

TABLE 1. Some examples of operators.

PDE system	$E(x)$	m_2
Laplace/ (nonlinear) Helmholtz	$\dfrac{-1}{2\pi}\ln r \ I$	1
linear elasticity/elastic waves	$\dfrac{-(\lambda + 3\mu)\ln r I + (\lambda + \mu)\mathbf{e_r}\mathbf{e_r}^T}{4\pi\mu(\lambda + 2\mu)}$	$\dfrac{\lambda + 3\mu}{2\mu(\lambda + 2\mu)}$
Stokes/quasi-Stokes, Navier-Stokes	$\dfrac{-\ln r I + \mathbf{e_r}\mathbf{e_r}^T}{4\pi\nu}$	$\dfrac{1}{2\nu}$

TABLE 2. Fundamental solution and coefficient m_2 (2D).

4.5. **Spherical hole (3D).** We suppose here that $\omega = B(0, 1)$ and that the fundamental solution of the operator $\tilde{\Delta}$ is of the form

$$E(x) = \frac{\alpha I + \beta \mathbf{e_r}\mathbf{e_r^T}}{4\pi r}, \qquad \alpha, \beta \in \mathbb{R}.$$

A straightforward calculus leads to the equality

$$\int_{\partial\omega} E(x - y)ds(y) = m_3 I \qquad \forall x \in \partial\omega,$$

$$m_3 = \alpha + \frac{\beta}{3}.$$

In this case, provided that $m_3 \neq 0$, the density η solution of (26) is the constant $\eta = m_3^{-1}X$ and the polarization matrix is

$$\mathcal{P}_{B(0,1)} = \frac{4\pi}{m_3}I.$$

Hence the topological asymptotic expansion reads

$$j(\rho) - j(0) \sim \rho\left[\frac{4\pi}{m_3}u_0(0).v_0(0) + \delta_{J1} + \delta_{J2}\right]. \tag{30}$$

Table 3 gathers the values of the coefficient m_3 corresponding to the operators presented in Table 1. Here and in Table 2, in the linear cases, we retrieve in a systematic way known formulas [5, 6, 7, 8, 12, 22, 23].

PDE system	$E(x)$	m_3
Laplace/ (nonlinear) Helmholtz	$\dfrac{1}{4\pi r} I$	1
linear elasticity/elastic waves	$\dfrac{(\lambda + 3\mu)I + (\lambda + \mu)\mathbf{e_r}\mathbf{e_r}^T}{8\pi\mu(\lambda + 2\mu)r}$	$\dfrac{2\lambda + 5\mu}{3\mu(\lambda + 2\mu)}$
Stokes/quasi-Stokes, Navier-Stokes	$\dfrac{I + \mathbf{e_r}\mathbf{e_r}^T}{8\pi\nu r}$	$\dfrac{2}{3\nu}$

TABLE 3. Fundamental solution and coefficient m_3 (3D).

4.6. Particular cost functionals.
The proof of the following theorem is reported in Section 5.

Theorem 4. *For the following cost functionals and an operator $\tilde{\Delta}$ satisfying Hypothesis 3, under Hypothesis 1 in 3D (resp. Hypothesis 2 in 2D), Equations (13) and (14) hold with $f(\rho) = \rho$ (resp. $f(\rho) = -1/\ln \rho$) and the corresponding values of δ_{J1} and δ_{J2}.*

(1) *If the cost functional is of the form*

$$J_\rho(u) \;=\; J(u_{|\Omega\setminus\overline{B(0,R)}}), \qquad R > 0,$$

then

$$\delta_{J1} = \delta_{J2} = 0.$$

(2) *For the cost functional*

$$J_\rho(u) \;=\; \int_{\Omega_\rho} |u - u_d|^2 dx$$

where $u_d \in L^2(\Omega)^n \cap L^p(B(0,R))^n, p > N, R > 0$, we have

$$\delta_{J1} = \delta_{J2} = 0.$$

(3) *For the cost functional*

$$J_\rho(u) \;=\; \int_{\Omega_\rho} |A\nabla(u - u_d)|^2 dx$$

where $u_d \in \mathcal{V}(\Omega) \cap W^{1,p}(B(0,R))^n, p > N, R > 0$, we have

$$\delta_{J1} = 0 \qquad and \qquad \delta_{J2} \;=\; \begin{cases} \mathcal{P}_\omega u_0(0).u_0(0) & in \; 3D, \\ \dfrac{2\pi}{m_2} u_0(0).u_0(0) & in \; 2D. \end{cases}$$

5. Proofs

In this section, we denote by R some fixed radius such that $\overline{B(0,R)} \subset \Omega$, $D_R = \Omega \setminus \overline{B(0,R)}$. We call c any positive number that may change from place to place but that never depends on ρ.

5.1. Proof of theorem 2 (3D).

5.1.1. *Error estimate on the solution.* We recall the following lemma [6].

Lemma 1. *Let w_ρ be the solution of*

$$\begin{cases} -\Delta w_\rho &= 0 & in & \mathbb{R}^3 \setminus \overline{\omega_\rho}, \\ w_\rho &\to 0 & at & \infty, \\ w_\rho &= \psi & on & \partial\omega_\rho, \end{cases}$$

where $\psi \in H^{1/2}(\partial\omega_\rho)$. There exists some constants $c > 0$ such that for all ρ sufficiently small

$$\begin{aligned}
\|w_\rho\|_{0,\Omega_\rho} &\leq c\rho\|\psi(\rho x)\|_{1/2,\partial\omega}, \\
\|w_\rho\|_{1,D_R} &\leq c\rho\|\psi(\rho x)\|_{1/2,\partial\omega}, \\
|w_\rho|_{1,\Omega_\rho} &\leq c\rho^{1/2}\|\psi(\rho x)\|_{1/2,\partial\omega}.
\end{aligned}$$

(1) *Estimate of h_ρ.* Lemma 1 yields

$$\begin{array}{llll}
\|h_\rho\|_{0,\Omega_\rho} &\leq c\rho\|u_0(\rho x)\|_{1/2,\partial\omega} &\leq c\rho, \\
\|h_\rho\|_{1,D_R} &\leq c\rho\|u_0(\rho x)\|_{1/2,\partial\omega} &\leq c\rho, \\
|h_\rho|_{1,\Omega_\rho} &\leq c\rho^{1/2}\|u_0(\rho x)\|_{1/2,\partial\omega} &\leq c\rho^{1/2}.
\end{array} \tag{31}$$

(2) *Estimate of r_ρ.* Thanks to Hypothesis 1, it comes

$$\|r_\rho\|_{1,\Omega_\rho} \leq c\|R_{u_0}(h_\rho)\|_{-1,\Omega_\rho} + c\|h_\rho\|_{1/2,\Gamma}.$$

Let $\tilde{h}_\rho$ be the extension

$$\tilde{h}_\rho = \begin{cases} h_\rho & \text{on} & \Omega_\rho, \\ u_0 & \text{on} & \omega_\rho. \end{cases}$$

Due to Equation (2) and Hypothesis 1, we have

$$\begin{aligned}
\|r_\rho\|_{1,\Omega_\rho} &\leq c\|R_{u_0}(\tilde{h}_\rho)\|_{-1,\Omega} + c\|h_\rho\|_{1/2,\Gamma} \\
&\leq c\|\tilde{h}_\rho\|_{0,\Omega} + c\|\tilde{h}_\rho\|^2_{1,\Omega} + c\|h_\rho\|_{1/2,\Gamma} \\
&\leq c\|h_\rho\|_{0,\Omega_\rho} + c\|u_0\|_{0,\omega_\rho} + c\|h_\rho\|^2_{1,\Omega_\rho} + c\|u_0\|^2_{1,\omega_\rho} + c\|h_\rho\|_{1/2,\Gamma}.
\end{aligned}$$

Then, Lemma 1 and the regularity of u_0 yield

$$\|r_\rho\|_{1,\Omega_\rho} \leq c\rho. \tag{32}$$

5.1.2. *Determination of δ_{F1}.* We have to prove that $|\mathcal{E}_i(\rho)| = o(\rho)$ for all $i = 1, ..., 4$.

(1) The regularity of u_0, v_0 and σ in the vicinity of the origin yields directly

$$|\mathcal{E}_1(\rho)| \leq c\rho^2.$$

(2) By a change of variable and thanks to the regularity of v_0 near the origin, it comes

$$|\mathcal{E}_2(\rho)| \leq c\rho\|\partial_n(r_\rho(\rho x))\|_{-1/2,\partial\omega}.$$

By difference, r_ρ is locally H^2 near the origin. Hence, with the notation

$$\tilde{H}_0^1(\frac{1}{\rho}\Omega_\rho) = \left\{ u \in H^1(\frac{1}{\rho}\Omega_\rho), u_{|\frac{1}{\rho}\Gamma} = 0 \right\},$$

we obtain

$$\begin{aligned}
|\mathcal{E}_2(\rho)| &\leq c\rho\left[|r_\rho(\rho x)|_{1,\frac{1}{\rho}\Omega_\rho} + \|\Delta(r_\rho(\rho x))\|_{\tilde{H}_0^1(\frac{1}{\rho}\Omega_\rho)'}\right] \\
&\leq c\rho|r_\rho(\rho x)|_{1,\frac{1}{\rho}\Omega_\rho} + c\rho^3\|(R_{u_0+h_\rho}(r_\rho) + R_{u_0}(h_\rho))(\rho x)\|_{\tilde{H}_0^1(\frac{1}{\rho}\Omega_\rho)'}.
\end{aligned}$$

Yet, $R_{u_0+h_\rho}(r_\rho) + R_{u_0}(h_\rho) = R_{u_0}(h_\rho + r_\rho)$ and Equation (2) yields

$$\|R_{u_0}(h_\rho + r_\rho)(\rho x)\|_{\tilde{H}_0^1(\frac{1}{\rho}\Omega_\rho)'} \leq c\|R_{u_0}(h_\rho + r_\rho)(\rho x)\|_{H_0^1(\frac{1}{\rho}\Omega)'} + c\|R_{u_0}(h_\rho + r_\rho)(\rho x)\|_{H^1(\omega)'}.$$

It follows from the fact that, inside ω_ρ, $h_\rho + r_\rho = -u_0$ which is of class $\mathcal{C}^2$, together with Hypothesis 1, that $\|R_{u_0}(h_\rho + r_\rho)(\rho x)\|_{H^1(\omega)'} \le c$. Then, a change of variable brings

$$|\mathcal{E}_2(\rho)| \le c\rho^{1/2}\|r_\rho\|_{1,\Omega_\rho} + c\rho^{1/2}\|R_{u_0}(h_\rho + r_\rho)\|_{H^{-1}(\Omega)} + c\rho^3.$$

By Hypothesis 1, we get

$$
\begin{aligned}
|\mathcal{E}_2(\rho)| \;\le\;& c\rho^{1/2}\|r_\rho\|_{1,\Omega_\rho} + c\rho^{1/2}(\|h_\rho + r_\rho\|_{0,\Omega} + \|h_\rho + r_\rho\|_{1,\Omega}^2) + c\rho^3 \\
\le\;& c\rho^{1/2}\|r_\rho\|_{1,\Omega_\rho} + c\rho^3 \\
& + c\rho^{1/2}(\|h_\rho + r_\rho\|_{0,\Omega_\rho} + \|h_\rho + r_\rho\|_{1,\Omega_\rho}^2 + \|u_0\|_{0,\omega_\rho} + \|u_0\|_{1,\omega_\rho}^2).
\end{aligned}
$$

Finally, the inequalities (31) and (32) and the regularity of u_0 imply

$$|\mathcal{E}_2(\rho)| \le c\rho^{3/2}.$$

(3) We have

$$|\mathcal{E}_3(\rho)| \le c\rho\|\partial_n S_\rho\|_{-1/2,\partial\omega} \le c\rho|S_\rho|_{1,B\setminus\overline{\omega}}$$

where B denotes some ball containing $\overline{\omega}$. By means of the elliptic regularity and a Taylor expansion of u_0 at the origin, we obtain that

$$\|S_\rho\|_{1,B\setminus\overline{\omega}} \le c\rho, \tag{33}$$

from which we derive

$$|\mathcal{E}_3(\rho)| \le c\rho^2.$$

(4) A Taylor expansion of v_0 yields straightforwardly

$$|\mathcal{E}_4(\rho)| \le c\rho^2.$$

5.1.3. *Determination of δ_{F2}.* By an immediate calculation, it comes

$$< F_0(u_\rho) - F_0(u_0) - DF_0(u_0)(u_\rho - u_0), v_0 > \; = \; < R_{u_0}(u_\rho - u_0) - D\Phi(u_0)(u_\rho - u_0), v_0 > .$$

Thanks to Hypothesis 1, and recalling that u_ρ is extended by zero inside ω_ρ, we obtain

$$
\begin{aligned}
& < F_0(u_\rho) - F_0(u_0) - DF_0(u_0)(u_\rho - u_0), v_0 > \\
& = o\left(\|u_\rho - u_0\|_{0,\Omega} + \|u_\rho - u_0\|_{1,\Omega}^2\right) \\
& = o\left(\|h_\rho + r_\rho\|_{0,\Omega_\rho} + \|u_0\|_{0,\omega_\rho} + \|h_\rho + r_\rho\|_{1,\Omega_\rho}^2 + \|u_0\|_{1,\omega_\rho}^2\right).
\end{aligned}
$$

From the inequalities (31) and (32) and the regularity of u_0 in the vicinity of the origin, we deduce that the left hand side behaves like a $o(\rho)$ and consequently that $\delta_{F2} = 0$.

5.2. **Proof of theorem 3 (2D).**

5.2.1. *Error estimate on the solution.* We recall the following lemma [6].

Lemma 2. *Let w_ρ be the solution of*

$$
\begin{cases}
-\Delta w_\rho & = & 0 & in & \Omega_\rho, \\
w_\rho & = & 0 & on & \Gamma, \\
w_\rho & = & \psi & on & \partial\omega_\rho.
\end{cases}
$$

where $\psi \in H^{1/2}(\partial\omega_\rho)$. There exists $c > 0$ such that for all ρ small enough,

$$\|w_\rho\|_{1,\Omega_\rho} \le \frac{c}{\sqrt{-\ln\rho}}\|\psi(\rho x)\|_{1/2,\partial\omega}.$$

(1) *Estimate of h_ρ.* Starting from the explicit expression of h_ρ, easy calculations yield

$$
\begin{aligned}
\|h_\rho\|_{W^{1,p}(\Omega)} &\le \frac{c}{-\ln\rho} & , & \qquad \|h_\rho\|_{1,\Omega_\rho} &\le \frac{c}{\sqrt{-\ln\rho}}, \\
\|h_\rho\|_{0,\Omega_\rho} &\le \frac{c}{-\ln\rho} & , & \qquad \|h_\rho\|_{1,D_R} &\le \frac{c}{-\ln\rho}.
\end{aligned}
\tag{34}
$$

(2) *Estimate of r_ρ.* Thanks to Hypothesis 2, we obtain

$$
\begin{aligned}
\|r_\rho\|_{1,\Omega} &\leq c\|R_{u_0}(h_\rho)\|_{-1,\Omega} + c\|h_\rho\|_{1/2,\Gamma} \\
&\leq c\|R_{u_0}(h_\rho)\|_{L^q(\Omega)} + c\|h_\rho\|_{1/2,\Gamma} \\
&\leq c\|h_\rho\|_{W^{1,p}(\Omega)} + c\|h_\rho\|_{1/2,\Gamma} \\
&\leq \frac{c}{-\ln\rho}.
\end{aligned}
\tag{35}
$$

Moreover, we have

$$
\Delta r_\rho = R_{u_0+h_\rho}(r_\rho) + R_{u_0}(h_\rho) = R_{u_0}(h_\rho + r_\rho).
$$

Thus, since $h_\rho + r_\rho \in W^{1,p}(\Omega)$, we have $R_{u_0}(h_\rho + r_\rho) \in L^q(\Omega)$. A standard interior regularity theorem yields $r_\rho \in W^{2,q}(B(0,R))$ and

$$
\|r_\rho\|_{W^{2,q}(B(0,R))} \leq c\|h_\rho + r_\rho\|_{W^{1,p}(\Omega)} \leq \frac{c}{-\ln\rho}.
$$

(3) *Estimate of s_ρ.* By splitting s_ρ into two functions and by estimating each of them with the help of Lemma 2 and Hypothesis 2, respectively, we prove easily that

$$
\|s_\rho\|_{1,\Omega_\rho} \leq \frac{c}{\sqrt{-\ln\rho}}\|(u_0 + h_\rho + r_\rho)(\rho x)\|_{1/2,\partial\omega}.
$$

On the one hand, we have on $\partial\omega$

$$
\begin{aligned}
(u_0 + h_\rho)(\rho x) &= u_0(\rho x) - \frac{E(\rho x)}{E(\rho)}u_0(0) \\
&= u_0(\rho x) - u_0(0) - \frac{E(x)}{E(\rho)}u_0(0),
\end{aligned}
$$

from which we derive straightforwardly

$$
\|(u_0 + h_\rho)(\rho x)\|_{1/2,\partial\omega} \leq \frac{c}{-\ln\rho}.
\tag{36}
$$

On the other hand, since $W^{2,q}(B(0,R)) \subset L^\infty(B(0,R))$, we have

$$
\begin{aligned}
\|r_\rho(\rho x)\|_{1/2,\partial\omega} &\leq c\|r_\rho(\rho x)\|_{1,B\setminus\overline{\omega}} \\
&\leq \frac{c}{\rho}\|r_\rho\|_{0,\rho B\setminus\overline{\omega_\rho}} + c|r_\rho|_{1,\rho B\setminus\overline{\omega_\rho}} \\
&\leq c\|r_\rho\|_{L^\infty(\Omega_\rho)} + c|r_\rho|_{1,\Omega_\rho} \\
&\leq c\|r_\rho\|_{W^{2,q}(\Omega_\rho)} \\
&\leq \frac{c}{-\ln\rho}.
\end{aligned}
$$

Finally, we obtain

$$
\|s_\rho\|_{1,\Omega_\rho} \leq \frac{c}{(-\ln\rho)^{3/2}}.
\tag{37}
$$

5.2.2. *Determination of δ_{F1}.*

(1) Thanks to the regularity of u_0, v_0 and σ near the origin, we obtain immediately that

$$
|\mathcal{E}_1(\rho)| \leq c\rho.
$$

(2) The Green formula provides

$$
\int_{\partial\omega_\rho} \partial_n r_\rho v_0 ds = \int_{\omega_\rho} (\Delta r_\rho v_0 + \nabla r_\rho.\nabla v_0)dx.
$$

Then, it follows from the Hölder inequality that

$$
\begin{aligned}
|\mathcal{E}_2(\rho)| &\leq c\|r_\rho\|_{W^{2,1}(\omega_\rho)} \\
&\leq c\rho^{2-2/q}\|r_\rho\|_{W^{2,q}(\omega_\rho)} \\
&\leq c\frac{\rho^{2-2/q}}{-\ln\rho} = o(\frac{-1}{\ln\rho}).
\end{aligned}
$$

(3) By the Green formula, we find that

$$
|\mathcal{E}_3(\rho)| \leq c|s_\rho|_{1,\Omega_\rho} + c\|\Delta s_\rho\|_{H^1(\Omega_\rho)'}.
$$

Yet, according to Hypothesis 2, we have

$$
\|\Delta s_\rho\|_{H^1(\Omega_\rho)'} = \|R_{u_0+h_\rho+r_\rho}(s_\rho)\|_{H^1(\Omega_\rho)'} \leq c\|s_\rho\|_{H^1(\Omega_\rho)}.
$$

Consequently, Equation (37) yields

$$
|\mathcal{E}_3(\rho)| \leq \frac{c}{(-\ln\rho)^{3/2}}.
$$

(4) We obtain directly from the definition of h_ρ and a Taylor expansion of v_0 that

$$
|\mathcal{E}_4(\rho)| \leq \frac{c\rho}{-\ln\rho}.
$$

5.2.3. *Determination of δ_{F2}.* We can prove that $\delta_{F2} = 0$ in a similar manner to the 3D case.

5.3. Proof of theorem 4 (particular cost functionals).

For simplicity, we present the proof for $\tilde{\Delta} = \Delta$. We need the following estimate which is a consequence of Equations (31) and (32) in 3D, of Equations (34), (35) and (37) in 2D.

Lemma 3. *For all ρ small enough, we have*

$$
\begin{aligned}
\|u_\rho - u_0\|_{1,D_R} &= O(f(\rho)), \\
\|u_\rho - u_0\|_{0,\Omega} &= O(f(\rho)).
\end{aligned}
$$

The proof of Theorem 4 is successively presented for the three examples of cost functional.

(1) The result comes straightforwardly from the differentiability of J and Lemma 3.

(2) On the one hand, we have

$$
J_\rho(u_\rho) - J_0(u_\rho) = -\int_{\omega_\rho} |u_d|^2 dx.
$$

From the Hölder inequality and the assumption made on the regularity of u_d, we obtain easily that

$$
J_\rho(u_\rho) - J_0(u_\rho) = o(f(\rho)).
$$

Hence $\delta_{J1} = 0$. On the other hand, we have

$$
J_0(u_\rho) - J_0(u_0) = DJ_0(u_0)(u_\rho - u_0) + \int_\Omega |u_\rho - u_0|^2 dx.
$$

According to Lemma 3, this latter term is a $o(f(\rho))$ and consequently $\delta_{J2} = 0$.

(3) We prove that $\delta_{J1} = 0$ in the same way as in the previous case. Besides, we have

$$
\begin{aligned}
V_J(u_\rho - u_0) &:= J_0(u_\rho) - J_0(u_0) - DJ_0(u_0)(u_\rho - u_0) \\
&= \int_\Omega |\nabla(u_\rho - u_0)|^2 dx.
\end{aligned}
$$

- Let us first consider the 3D case. With the splitting (16), we have

$$V_J(u_\rho - u_0) = \int_{\Omega_\rho} |\nabla(h_\rho + r_\rho)|^2 dx + \int_{\omega_\rho} |\nabla u_0|^2 dx.$$

Thanks to the boundedness of ∇u_0, the latter term is a $o(\rho)$. It follows from the estimates (31) and (32) that

$$V_J(u_\rho - u_0) = \int_{\Omega_\rho} |\nabla h_\rho|^2 dx + o(\rho).$$

The Green formula, a change of variable, the relation $H_\rho = H + S_\rho$ and the estimate (33) bring successively

$$\begin{aligned}
V_J(u_\rho - u_0) &= -\int_{\partial\omega_\rho} \partial_n h_\rho . h_\rho ds + o(\rho) \\
&= -\rho \int_{\partial\omega} \partial_n H_\rho . H_\rho ds + o(\rho) \\
&= -\rho \int_{\partial\omega} \partial_n H . H ds + o(\rho).
\end{aligned}$$

Then, the jump relation of the single layer potential yields

$$V_J(u_\rho - u_0) = -\rho \int_{\partial\omega} \eta . u_0(0) ds + o(\rho),$$

from which we deduce the expression of δ_{J2}.

- In 2D, we have according to the splitting (23)

$$V_J(u_\rho - u_0) = \int_{\Omega_\rho} |\nabla(h_\rho + r_\rho + s_\rho)|^2 dx + \int_{\omega_\rho} |\nabla u_0|^2 dx.$$

We derive from Equations (34), (35) and (37)

$$\begin{aligned}
V_J(u_\rho - u_0) &= \int_{\Omega_\rho} |\nabla h_\rho|^2 dx + o(\frac{-1}{\ln \rho}) \\
&= -\int_{\partial\omega_\rho} \partial_n h_\rho . h_\rho ds + o(\frac{-1}{\ln \rho}).
\end{aligned}$$

Next, it comes from (36) and the regularity of u_0 in the vicinity of the origin that

$$V_J(u_\rho - u_0) = \left(\int_{\partial\omega_\rho} \partial_n h_\rho ds \right) u_0(0) + o(\frac{-1}{\ln \rho}).$$

We obtain finally by replacing h_ρ by its expression and by using the fact that E is the fundamental solution of the Laplacian that

$$V_J(u_\rho - u_0) = \frac{u_0(0)^2}{E(\rho)} + o(\frac{-1}{\ln \rho}).$$

This leads to the announced value of δ_{J2}.

6. An example of application: a nonlinear Helmholtz equation

Theorem 5. *We consider the function*

$$\Phi(u) = -k^2(1 + \epsilon|u|^{2s})u,$$

where $k \in \mathbb{C}$, $\epsilon \in \mathbb{C}$ and $s \in \{1, 2\}$ in 3D, $s \in \mathbb{N}^*$ in 2D. We assume that k^2 is not an eigenvalue of the operator $-\Delta : H_0^1(\Omega) \to H^{-1}(\Omega)$ and that the direct and adjoint states u_0 and v_0 belong to $L^\infty(\Omega)$. Then,

- *for all open set $\mathcal{O} \subset \Omega$, Φ defines a differentiable map from $H^1(\mathcal{O})$ into $H^1(\mathcal{O})'$ and Equation (2) holds,*
- *Hypothesis 1 in 3D and Hypothesis 2 in 2D are satisfied.*

Those results remain true in the case of a vector field with an operator $\tilde{\Delta}$ verifying Hypothesis 3.

Proof. For simplicity, we present the proof for a real scalar field (*i.e.* k, ϵ and σ are real), in 3D only. The 2D case can be treated in a similar manner.

(1) The fact that Φ maps a function of $H^1(\mathcal{O})$ to a function of $H^1(\mathcal{O})'$ and that Equation (2) is satisfied is an immediate consequence of the Sobolev imbedding $H^1(\mathcal{O}) \subset L^6(\mathcal{O})$ and of the Hölder inequality.

(2) Let us now study the operator $R_u : v \mapsto \Phi(u+v) - \Phi(u)$. The binomial formula yields

$$R_u(v) = -k^2[1 + \epsilon(2s+1)u^{2s}]v + R'_u(v),$$

with

$$R'_u(v) = -k^2 \epsilon v^2 \sum_{p=0}^{2s-1} \binom{2s+1}{p+2} u^{2s-1-p} v^p.$$

For a given $u \in H^1(\mathcal{O})$, we obtain from the Sobolev imbedding theorem and the Hölder inequality that

- the map $v \mapsto -k^2[1 + \epsilon(2s+1)u^{2s}]v$ is linear and continuous from $H^1(\mathcal{O})$ into $H^1(\mathcal{O})'$,
- the estimate

$$\|R'_u(v)\|_{H^1(\mathcal{O})'} \leq c\|v\|^2_{H^1(\mathcal{O})}$$

holds true provided that $\|v\|_{H^1(\mathcal{O})}$ is small enough.

Therefore, we conclude that Φ is differentiable from $H^1(\mathcal{O})$ into $H^1(\mathcal{O})'$ with

$$D\Phi(u)v = -k^2[1 + \epsilon(2s+1)u^{2s}]v.$$

Consider now the case where $\mathcal{O} = \Omega$ and $u = u_0$, the direct state. As $u_0 \in L^\infty(\Omega)$, we have on the one hand

$$\|D\Phi(u_0)v\|_{H^1(\Omega)'} \leq c\|v\|_{L^2(\Omega)},$$

and consequently,

$$\|R_{u_0}(v)\|_{H^1(\Omega)'} \leq c(\|v\|_{L^2(\Omega)} + \|v\|^2_{H^1(\Omega)}).$$

On the other hand, since furthermore $v_0 \in L^\infty(\Omega)$, we have for all $v \in H^1(\Omega)$

$$
\begin{aligned}
| < R_{u_0}(v) - DR_{u_0}(0)v, v_0 > | &= | < R'_{u_0}(v), v_0 > | \\
&\leq c\|R'_{u_0}(v)\|_{L^1(\Omega)} \\
&\leq c \left\| v^2 + \sum_{p=1}^{2s-1} \binom{2s+1}{p+2} u_0^{2s-1-p} v^{p+2} \right\|_{L^1(\Omega)}.
\end{aligned}
$$

For $\|v\|_{H^1(\Omega)}$ small enough, the Sobolev imbedding theorem and the Hölder inequality yield

$$
\begin{aligned}
| < R_{u_0}(v) - DR_{u_0}(0)v, v_0 > | &\leq c(\|v\|^2_{L^2(\Omega)} + \|v\|^3_{H^1(\Omega)}) \\
&\leq c\|v\|_{H^1(\Omega)}(\|v\|_{L^2(\Omega)} + \|v\|^2_{H^1(\Omega)}).
\end{aligned}
$$

(3) We will now sketch the checking of the first condition of Hypothesis 1, which uses very standard arguments. For a given $\varphi \in H^{1/2}(\Gamma)$ and any $\psi \in H^{-1}(\Omega_\rho)$, we denote by $S_\rho^\varphi(\psi)$ the solution of the PDE

$$\left\{ \begin{array}{rclcl} -\Delta y - k^2 y & = & \psi & \text{in} & \Omega_\rho, \\ y & = & \varphi & \text{on} & \Gamma, \\ y & = & 0 & \text{on} & \partial\omega_\rho. \end{array} \right.$$

By elliptic regularity, we have

$$\|S_\rho^\varphi(\psi)\|_{1,\Omega_\rho} \leq c(\|\psi\|_{-1,\Omega_\rho} + \|\varphi\|_{1/2,\Gamma}). \tag{38}$$

Moreover, provided that ρ is small enough, this constant c can be taken independent of ρ (see e.g. [1]). Therefore, Problem (20) can be rewritten

$$T_\rho^{f,\varphi}(v) = v, \tag{39}$$

where

$$T_\rho^{f,\varphi}(v) = S_\rho^\varphi \left(k^2 \epsilon (2s+1) u^{2s} v - R'_u(v) + f \right).$$

Using (38) and a few technical estimates coming basically from the Hölder inequality, it is easy to show that, when $\|f\|_{-1,\Omega_\rho}$, $\|\varphi\|_{1/2,\Gamma}$ and $\|u\|_{1,\Omega_\rho}$ are sufficiently small, there exists $\beta > 0$ such that the ball $B_{H^1(\Omega_\rho)}(0,\beta)$ is stable by the map $T_\rho^{f,\varphi}$ and such that $T_\rho^{f,\varphi}$ is a 1/2-contraction on $B_{H^1(\Omega_\rho)}(0,\beta)$. Hence, by the Banach fixed point theorem, Equation (39) admits one and only one solution. Finally, to obtain the elliptic regularity of the solution v, we write that

$$\|T_\rho^{f,\varphi}(v) - T_\rho^{f,\varphi}(0)\|_{1,\Omega_\rho} \leq \frac{1}{2}\|v - 0\|_{1,\Omega_\rho},$$

because $T_\rho^{f,\varphi}$ is a 1/2-contraction. This implies, since v is a fixed point, that

$$\|v - S_\rho^\varphi(f)\|_{1,\Omega_\rho} \leq \frac{1}{2}\|v\|_{1,\Omega_\rho}.$$

Thus, using also (38), it comes

$$\|v\|_{1,\Omega_\rho} \leq 2\|S_\rho^\varphi(f)\|_{1,\Omega_\rho} \leq c(\|f\|_{-1,\Omega_\rho} + \|\varphi\|_{1/2,\Gamma}).$$

This completes the proof of Theorem 5.

$\square$

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