

Sensitivity analysis with respect to a local perturbation of the material property

Samuel Amstutz

Johann Radon Institute for Computational and Applied Mathematics (RICAM),
Austrian Academy of Sciences,
Altenbergerstrasse 69, A-4040 Linz, Austria.
email: samuel.amstutz@oeaw.ac.at

Abstract

In the present work, the notion of topological sensitivity is extended to the case of a local perturbation of the properties of the material constitutive of the domain. As a model example, we consider the problem

$$-\operatorname{div}(\alpha_\varepsilon A \nabla u_\varepsilon) + \beta_\varepsilon u_\varepsilon = F_\varepsilon$$

in two and three dimensions, where A is a symmetric positive definite matrix and $\alpha_\varepsilon, \beta_\varepsilon, F_\varepsilon$ are functions whose values inside a small subdomain ω_ε are different from those of the background medium. An adjoint method is used to determine an asymptotic expansion of a given criterion when the diameter of ω_ε goes to zero.

Key-words and phrases: topology optimization, topological sensitivity analysis, detection of inclusions.

1 Introduction

In the last few years, the notion of topological sensitivity has become increasingly widespread in the shape optimization community. In contrast to the classical techniques of boundary variation, this tool, among some others like homogenization or level-set based methods, allows to deal with problems for which the topology (*i.e.* the number of holes) of the optimal domain is a priori unknown. The principle consists in studying directly the behavior of the shape functional of interest when creating a small hole inside the domain. From the mathematical point of view, given a criterion $\mathcal{J}(\Omega)$, $\Omega \in \mathbb{R}^d$ ($d=2$ or 3), a point $x_0 \in \Omega$ and a fixed domain $\omega \subset \mathbb{R}^d$, one searches for an asymptotic expansion of the form

$$\mathcal{J}(\Omega \setminus \overline{(x_0 + \varepsilon\omega)}) - \mathcal{J}(\Omega) = f(\varepsilon)g(x_0) + o(f(\varepsilon)), \quad (1.1)$$

where $f(\varepsilon)$ is an explicit positive function going to zero with ε . The function g is commonly called “topological gradient” or “topological derivative”, and (1.1) the “topological asymptotic expansion”. Therefore, to minimize the criterion \mathcal{J} , one has to create holes at some points where the topological gradient is negative. This approach was instigated by Schumacher *et al.* [22], and then developed by many authors. For more details about the mathematical aspects and the related numerical procedures, the reader is referred *e.g.* to the publications [23, 16, 13, 10, 14, 18, 8].

Another situation, firstly addressed by Cedio-Fengya *et al.* [11], consists in studying the influence of the insertion of a small inhomogeneity which is nonempty, but whose constitutive parameters are different from those of the background medium. Other references on this topic can be found *e.g.* in [5, 4, 3, 2, 7, 1]. These works present two major differences with the previous ones. First, an interface condition holds on the border of the inclusion, instead of a classical boundary condition (usually of Dirichlet or Neumann type). Second, the authors being merely concerned by identification problems by means of boundary measurements, they provide asymptotic formulas either for objective functions specifically designed to this purpose, or of the solution itself at the location of the sensors. Therefore an adjoint state is not (at least explicitly) involved.

In the present work, the link between the two above approaches is investigated. A Helmholtz type state equation with an inhomogeneity in the coefficients is considered. For a certain class of objective functions, an asymptotic expansion of the form (1.1) is derived with the help of a relevant adjoint method. Unsurprisingly, a polarization matrix comes in to characterize the shape of the inhomogeneity, which enables to benefit from the numerous properties known about them. In particular, this leads to an explicit formulation of $g(x_0)$ in many practical cases. At the limit where the material parameters of the inhomogeneity tend to zero, the usual topological gradient for a hole with Neumann boundary condition is rigorously retrieved [13, 8].

The paper is organized as follows. The adjoint method is presented in an abstract framework in Section 2. Then, for simplicity, the scalar problem is first addressed. It is described in Section 3, its asymptotic analysis is carried out in Sections 4 and 5, and the obtained formula is stated and commented in Section 6. For the sake of readability, the proofs of all intermediate estimates are reported in Section 9. Some examples of cost functions are exhibited in Section 7. In Section 8, these results are generalized to the vector case, with the linear elasticity system given as an example. Section 10 is devoted to some numerical experiments.

2 A preliminary result

The following proposition describes in an abstract framework the adjoint method we will use to derive the first variation of a given cost function. For the sake of simplicity, real scalar fields are considered in all the analysis. The adaptation to the complex and vector cases presents no conceptual difficulty (see *e.g.* [20, 8] to figure out the peculiarities of the complex case).

Proposition 2.1. *Let \mathcal{V} be a real Hilbert space. For all parameter $\varepsilon \in [0, \varepsilon_0[$, $\varepsilon_0 > 0$, consider a vector $u_\varepsilon \in \mathcal{V}$ solving a variational problem of the form*

$$a_\varepsilon(u_\varepsilon, v) = \ell_\varepsilon(v) \quad \forall v \in \mathcal{V}, \quad (2.1)$$

where a_ε and ℓ_ε are a bilinear form on \mathcal{V} and a linear form on \mathcal{V} , respectively. Consider now a cost function

$$j(\varepsilon) = J_\varepsilon(u_\varepsilon) \in \mathbb{R} \quad (2.2)$$

where, for $\varepsilon \in [0, \varepsilon_0[$, the functional $J_\varepsilon : \mathcal{V} \rightarrow \mathbb{R}$ is Fréchet-differentiable at the point u_0 . Suppose that the following hypotheses hold.

1. There exist two numbers δa and $\delta \ell$ and a function $f(\varepsilon) \geq 0$ such that, when ε goes to zero,

$$(a_\varepsilon - a_0)(u_0, v_\varepsilon) = f(\varepsilon)\delta a + o(f(\varepsilon)), \quad (2.3)$$

$$(\ell_\varepsilon - \ell_0)(v_\varepsilon) = f(\varepsilon)\delta \ell + o(f(\varepsilon)), \quad (2.4)$$

$$\lim_{\varepsilon \rightarrow 0} f(\varepsilon) = 0, \quad (2.5)$$

where $v_\varepsilon \in \mathcal{V}$ is an adjoint state satisfying

$$a_\varepsilon(\varphi, v_\varepsilon) = -DJ_\varepsilon(u_0)\varphi \quad \forall \varphi \in \mathcal{V}. \quad (2.6)$$

2. There exist two numbers δJ_1 and δJ_2 such that

$$J_\varepsilon(u_\varepsilon) = J_\varepsilon(u_0) + DJ_\varepsilon(u_0)(u_\varepsilon - u_0) + f(\varepsilon)\delta J_1 + o(f(\varepsilon)), \quad (2.7)$$

$$J_\varepsilon(u_0) = J_0(u_0) + f(\varepsilon)\delta J_2 + o(f(\varepsilon)). \quad (2.8)$$

Then the first variation of the cost function with respect to ε is given by

$$j(\varepsilon) - j(0) = f(\varepsilon)(\delta a - \delta \ell + \delta J_1 + \delta J_2) + o(f(\varepsilon)).$$

Proof. We have due to (2.1)

$$j(\varepsilon) - j(0) = [J_\varepsilon(u_\varepsilon) - J_0(u_0)] + [a_\varepsilon(u_\varepsilon, v_\varepsilon) - a_0(u_0, v_\varepsilon)] - [\ell_\varepsilon(v_\varepsilon) - \ell_0(v_\varepsilon)].$$

Using Equations (2.3) and (2.4) it comes

$$j(\varepsilon) - j(0) = J_\varepsilon(u_\varepsilon) - J_0(u_0) + a_\varepsilon(u_\varepsilon - u_0, v_\varepsilon) + f(\varepsilon)(\delta a - \delta \ell) + o(f(\varepsilon)).$$

It follows from (2.7) and (2.8) that

$$j(\varepsilon) - j(0) = DJ_\varepsilon(u_0)(u_\varepsilon - u_0) + a_\varepsilon(u_\varepsilon - u_0, v_\varepsilon) + f(\varepsilon)(\delta J_1 + \delta J_2 + \delta a - \delta \ell) + o(f(\varepsilon)).$$

The adjoint equation (2.6) leads to the announced result. \square

3 Problem formulation

Let Ω be a bounded subdomain of \mathbb{R}^d , $d = 2$ or 3 . For simplicity we assume that the boundary $\partial\Omega$ is of class \mathcal{C}^∞ , but this condition could be considerably weakened. We consider a small subdomain $\omega_\varepsilon \subset \Omega$ of the form $\omega_\varepsilon = x_0 + \varepsilon\omega$, where $x_0 \in \Omega$ and $\omega \subset \mathbb{R}^d$ is a bounded, smooth (\mathcal{C}^∞) domain containing the origin.

Let A be a symmetric positive definite matrix and $\alpha_0, \alpha_1, \beta_0, \beta_1$ be real numbers. For every small parameter $\varepsilon \geq 0$ (*i.e.* ε is lower than some $\varepsilon_0 > 0$), consider the piecewise constant coefficients

$$\alpha_\varepsilon(x) = \begin{cases} \alpha_0 & \text{if } x \in \Omega \setminus \overline{\omega_\varepsilon}, \\ \alpha_1 & \text{if } x \in \omega_\varepsilon, \end{cases} \quad \beta_\varepsilon(x) = \begin{cases} \beta_0 & \text{if } x \in \Omega \setminus \overline{\omega_\varepsilon}, \\ \beta_1 & \text{if } x \in \omega_\varepsilon, \end{cases} \quad (3.1)$$

and, given $F_0, F_1 \in H^2(\Omega)$, the function

$$F_\varepsilon = \begin{cases} F_0 & \text{in } \Omega \setminus \overline{\omega_\varepsilon}, \\ F_1 & \text{in } \omega_\varepsilon. \end{cases}$$

Moreover, we assume that F_0 and F_1 are Lipschitz continuous in the vicinity of the origin. We shall investigate two cases.

1. *First case:* $\alpha_0 > 0, \alpha_1 > 0, \beta_0\beta_1 \geq 0$. We consider a function $u_\varepsilon \in H^1(\Omega)$ solving the boundary value problem

$$\begin{cases} -\operatorname{div}(\alpha_\varepsilon A \nabla u_\varepsilon) + \beta_\varepsilon u_\varepsilon = F_\varepsilon & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases} \quad (3.2)$$

2. *Second case:* $\alpha_0 > 0, \alpha_1 = \beta_1 = 0, F_1 = 0$. In this extreme situation we are concerned by the PDE

$$\begin{cases} -\operatorname{div}(\alpha_0 A \nabla u_\varepsilon) + \beta_0 u_\varepsilon = F_0 & \text{in } \Omega \setminus \overline{\omega_\varepsilon}, \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \\ A \nabla u_\varepsilon \cdot n = 0 & \text{on } \partial\omega_\varepsilon. \end{cases} \quad (3.3)$$

In order to define the function u_ε in the whole domain Ω , the above system is complemented by the following

$$\begin{cases} -\operatorname{div}(A \nabla u_\varepsilon) = 0 & \text{in } \omega_\varepsilon, \\ u_\varepsilon \in H^1(\Omega). \end{cases} \quad (3.4)$$

We assume that the data are chosen in such a way that Problem (3.2) (resp. Problem (3.3)) admits one and only one solution for every $\varepsilon < \varepsilon_0$. It is worth mentioning that the Dirichlet condition on $\partial\Omega$ has been chosen merely to fix ideas, but it could be replaced by any other linear boundary condition provided that the problem of interest remains well-posed.

In both cases, u_ε satisfies the variational equality (2.1) in the space

$$\mathcal{V} = H_0^1(\Omega)$$

with the bilinear and linear forms

$$a_\varepsilon(u, v) = \int_\Omega \alpha_\varepsilon A \nabla u \cdot \nabla v \, dx + \int_\Omega \beta_\varepsilon uv \, dx, \quad (3.5)$$

$$\ell_\varepsilon(v) = \int_\Omega F_\varepsilon v \, dx. \quad (3.6)$$

Therefore, we will be able to carry out a single asymptotic analysis valid for the two cases. Nevertheless a separate study will be needed for a few intermediate results. We underline that Proposition 2.1 does not require the unique solvability of (2.1). This situation occurs in the second case, where the solution is fixed by imposing the additional condition (3.4). The possible dependence of the result on this choice of the solution is contained in the terms δJ_1 and δJ_2 (see comment below).

To enter into the framework of Proposition 2.1, we consider a cost function of the form (2.2) where the functional $J_\varepsilon : H_0^1(\Omega) \rightarrow \mathbb{R}$ is Fréchet-differentiable at the point u_0 and satisfies the conditions (2.7) and (2.8) for $f(\varepsilon) = \varepsilon^d$. We suppose furthermore that

$$\|DJ_\varepsilon(u_0) - DJ_0(u_0)\|_{-1, \Omega} = o(\varepsilon^{d/2}), \quad (3.7)$$

and that $DJ_0(u_0) \in H^2(\Omega)$. In the second case, we consider cost functionals which do not involve the value of the solution inside ω_ε , so that the extension (3.4), introduced for the needs of the analysis, plays no role in the final asymptotics. The reader is referred to Section 7 for some examples.

Let us now proceed to the checking of the hypotheses of Proposition 2.1 for the problem described above. Note that the straightforward generalization of the standard adjoint

method to the calculus of an asymptotic expansion is not applicable here, even in the first case, unless a trick like a truncation technique is used (see [16, 13]). Indeed, an estimate of the form

$$(a_\varepsilon - a_0)(u, v) = |\omega_\varepsilon| [(\alpha_1 - \alpha_0)A\nabla u(x_0) \cdot \nabla v(x_0) + (\beta_1 + \beta_0)u(x_0)v(x_0)] + o(|\omega_\varepsilon|)$$

would be needed for every $u, v \in H_0^1(\Omega)$, whereas it obviously holds only for smooth functions.

4 Variation of the bilinear form

In this section we concentrate on the asymptotic analysis of the variation

$$(a_\varepsilon - a_0)(u_0, v_\varepsilon) = \int_{\omega_\varepsilon} [(\alpha_1 - \alpha_0)A\nabla u_0 \cdot \nabla v_\varepsilon + (\beta_1 - \beta_0)u_0 v_\varepsilon] dx. \quad (4.1)$$

Let us first look into the behavior of the adjoint state v_ε . The associated PDE depends on the case under consideration.

1. *First case.* The classical formulation of the PDE associated to (2.6) reads

$$\begin{cases} -\operatorname{div}(\alpha_\varepsilon A\nabla v_\varepsilon) + \beta_\varepsilon v_\varepsilon &= -DJ_\varepsilon(u_0) & \text{in } \Omega, \\ v_\varepsilon &= 0 & \text{on } \partial\Omega, \end{cases} \quad (4.2)$$

which has one and only one solution in our context.

2. *Second case.* Since in this case Equation (2.6) does not provide an unique solution, we choose a particular one by enforcing

$$\begin{cases} -\operatorname{div}(\alpha_0 A\nabla v_\varepsilon) + \beta_0 v_\varepsilon &= -DJ_\varepsilon(u_0) & \text{in } \Omega \setminus \overline{\omega_\varepsilon}, \\ v_\varepsilon &= 0 & \text{on } \partial\Omega, \\ (A\nabla v_\varepsilon \cdot n)^- &= 0 & \text{on } \partial\omega_\varepsilon, \\ v_\varepsilon^+ &= v_\varepsilon^- & \text{on } \partial\omega_\varepsilon^-, \\ -\operatorname{div}(A\nabla v_\varepsilon) &= 0 & \text{in } \omega_\varepsilon. \end{cases} \quad (4.3)$$

The superscripts $+$ and $-$ indicate that $\partial\omega$ is approached from inside and outside, respectively.

By splitting in (4.1) v_ε into $v_\varepsilon = v_0 + (v_\varepsilon - v_0)$ and by introducing the ‘‘small’’ terms (this statement will be explained and checked later on)

$$\mathcal{E}_1(\varepsilon) = \int_{\omega_\varepsilon} [(\alpha_1 - \alpha_0)(A\nabla u_0 \cdot \nabla v_0 - A\nabla u_0(x_0) \cdot \nabla v_0(x_0)) + (\beta_1 - \beta_0)(u_0 v_0 - u_0(x_0)v_0(x_0))] dx,$$

$$\mathcal{E}_2(\varepsilon) = \int_{\omega_\varepsilon} (\beta_1 - \beta_0)u_0(v_\varepsilon - v_0) dx,$$

we obtain

$$(a_\varepsilon - a_0)(u_0, v_\varepsilon) = \varepsilon^d |\omega| [(\alpha_1 - \alpha_0)A\nabla u_0(x_0) \cdot \nabla v_0(x_0) + (\beta_1 - \beta_0)u_0(x_0)v_0(x_0)] + \mathcal{F}(\varepsilon) + \mathcal{E}_1(\varepsilon) + \mathcal{E}_2(\varepsilon). \quad (4.4)$$

We have isolated for convenience the term

$$\mathcal{F}(\varepsilon) = (\alpha_1 - \alpha_0) \int_{\omega_\varepsilon} A\nabla u_0 \cdot \nabla (v_\varepsilon - v_0) dx,$$

and we will now study its asymptotic behavior. To begin with, we approximate the variation $v_\varepsilon - v_0$ by the function

$$h_\varepsilon(x) = -\varepsilon(\alpha_1 - \alpha_0)H\left(\frac{x - x_0}{\varepsilon}\right), \quad (4.5)$$

where the function H (independent of ε) is the unique solution of

$$\left\{ \begin{array}{lll} -\operatorname{div}(A\nabla H) = 0 & \text{in } \omega \cup (\mathbb{R}^d \setminus \bar{\omega}), & (i) \\ H^+ - H^- = 0 & \text{on } \partial\omega, & (ii) \\ \alpha_1(A\nabla H.n)^+ - \alpha_0(A\nabla H.n)^- = A\nabla v_0(0).n & \text{on } \partial\omega, & (iii) \\ H \rightarrow 0 & \text{at } \infty. & (iv) \end{array} \right. \quad (4.6)$$

Therefore we write

$$\mathcal{F}(\varepsilon) = (\alpha_1 - \alpha_0) \int_{\omega_\varepsilon} A\nabla u_0 \cdot \nabla h_\varepsilon dx + \mathcal{E}_3(\varepsilon),$$

with

$$\mathcal{E}_3(\varepsilon) = (\alpha_1 - \alpha_0) \int_{\omega_\varepsilon} A\nabla u_0 \cdot \nabla (v_\varepsilon - v_0 - h_\varepsilon) dx.$$

The symmetry of A , the Green formula, and a change of variable yield successively

$$\begin{aligned} \mathcal{F}(\varepsilon) &= (\alpha_1 - \alpha_0) \int_{\omega_\varepsilon} \nabla(u_0 - u_0(x_0)) \cdot A\nabla h_\varepsilon dx + \mathcal{E}_3(\varepsilon) \\ &= (\alpha_1 - \alpha_0) \int_{\partial\omega_\varepsilon} (u_0 - u_0(x_0))(A\nabla h_\varepsilon \cdot n)^+ ds + \mathcal{E}_3(\varepsilon) \\ &= -\varepsilon^{d-1}(\alpha_1 - \alpha_0)^2 \int_{\partial\omega} (u_0(x_0 + \varepsilon y) - u_0(x_0))(A\nabla H(y) \cdot n(y))^+ ds(y) + \mathcal{E}_3(\varepsilon). \end{aligned}$$

Then, by setting

$$\mathcal{E}_4(\varepsilon) = -\varepsilon^{d-1}(\alpha_1 - \alpha_0)^2 \int_{\partial\omega} (u_0(x_0 + \varepsilon y) - u_0(x_0) - \nabla u_0(x_0) \cdot \varepsilon y)(A\nabla H(y) \cdot n(y))^+ ds(y),$$

we obtain

$$\begin{aligned} \mathcal{F}(\varepsilon) &= -\varepsilon^d(\alpha_1 - \alpha_0)^2 \int_{\partial\omega} (\nabla u_0(x_0) \cdot y)(A\nabla H(y) \cdot n(y))^+ ds(y) + \mathcal{E}_3(\varepsilon) + \mathcal{E}_4(\varepsilon) \\ &= -\varepsilon^d(\alpha_1 - \alpha_0)^2 \nabla u_0(x_0) \cdot \left[\int_{\partial\omega} (A\nabla H(y) \cdot n(y))^+ y ds(y) \right] + \mathcal{E}_3(\varepsilon) + \mathcal{E}_4(\varepsilon). \end{aligned}$$

Since the function H is continuous across $\partial\omega$, it can be represented with the help of a single layer potential (see *e.g.* [12]), namely there exists $p \in H^{-1/2}(\partial\omega)$ such that

$$\int_{\partial\omega} p dx = 0, \quad (4.7)$$

$$H(x) = \int_{\partial\omega} \frac{p(y)}{\alpha_1 - \alpha_0} E(x - y) ds(y), \quad (4.8)$$

where E denotes the fundamental solution of the operator $u \mapsto -\operatorname{div}(A\nabla u)$. The division of the density by $\alpha_1 - \alpha_0$ is meant to simplify some forthcoming expressions. The trivial case $\alpha_1 = \alpha_0$, for which $\mathcal{F}(\varepsilon) = 0$, is excluded until the end of this section. It follows from the jump relation

$$(A\nabla H.n)^+ - (A\nabla H.n)^- = \frac{p}{\alpha_1 - \alpha_0}$$

together with (4.6 iii) that

$$(\alpha_1 - \alpha_0)(A\nabla H.n)^+ = -\frac{\alpha_0}{\alpha_1 - \alpha_0} p + A\nabla v_0(x_0).n.$$

Hence

$$\mathcal{F}(\varepsilon) = \varepsilon^d(\alpha_1 - \alpha_0) \nabla u_0(x_0) \cdot \left[\int_{\partial\omega} \left(\frac{\alpha_0}{\alpha_1 - \alpha_0} p - A\nabla v_0(x_0).n \right) x ds \right] + \mathcal{E}_3(\varepsilon) + \mathcal{E}_4(\varepsilon). \quad (4.9)$$

To compute the density p , we replace in (4.6 iii) the normal derivatives by their expressions

$$(\alpha_1 - \alpha_0)(A\nabla H(x).n(x))^\pm = \pm \frac{p(x)}{2} + \int_{\partial\omega} p(y)(A\nabla E(x-y).n(x))ds(y).$$

This leads to the integral equation

$$\frac{\alpha_1 + \alpha_0}{\alpha_1 - \alpha_0} \frac{p(x)}{2} + \int_{\partial\omega} p(y)A\nabla E(x-y).n(x)ds(y) = A\nabla v_0(x_0).n(x) \quad \forall x \in \partial\omega. \quad (4.10)$$

According to the classical theory of integral equations of the second kind, Equation (4.10) admits one and only one solution $p \in H^{-1/2}(\partial\omega)$. Moreover, by linearity, there exists a $d \times d$ matrix $\mathcal{P}_{\omega, \alpha_1/\alpha_0}$ such that

$$\int_{\partial\omega} pxds = \mathcal{P}_{\omega, \alpha_1/\alpha_0} \nabla v_0(x_0). \quad (4.11)$$

Besides, an integration by parts provides

$$\int_{\partial\omega} xn^T ds = |\omega|I, \quad (4.12)$$

where I is the identity matrix. Gathering (4.4), (4.9), (4.11) and (4.12), we get

$$(a_\varepsilon - a_0)(u_0, v_\varepsilon) = \varepsilon^d [\alpha_0 \nabla u_0(x_0)^T \mathcal{P}_{\omega, \alpha_1/\alpha_0} \nabla v_0(x_0) + (\beta_1 - \beta_0)|\omega|u_0(x_0)v_0(x_0)] + \sum_{i=1}^4 \mathcal{E}_i(\varepsilon).$$

We prove in Section 9 that $|\mathcal{E}_i(\varepsilon)| = o(\varepsilon^d)$ for all $i = 1, \dots, 4$. Therefore Equations (2.3) and (2.5) hold with

$$\delta a = \alpha_0 \nabla u_0(x_0)^T \mathcal{P}_{\omega, \alpha_1/\alpha_0} \nabla v_0(x_0) + (\beta_1 - \beta_0)|\omega|u_0(x_0)v_0(x_0),$$

5 Variation of the linear form

Let us now turn to the variation

$$(\ell_\varepsilon - \ell_0)(v_\varepsilon) = \int_{\omega_\varepsilon} (F_1 - F_0)v_\varepsilon dx.$$

The regularity assumptions on F_0 and F_1 allow us to rewrite this expression in the form

$$(\ell_\varepsilon - \ell_0)(v_\varepsilon) = \varepsilon^d |\omega|(F_1 - F_0)(x_0)v_0(x_0) + \sum_{i=5}^6 \mathcal{E}_i(\varepsilon)$$

with

$$\begin{aligned} \mathcal{E}_5(\varepsilon) &= \int_{\omega_\varepsilon} (F_1 - F_0)(v_\varepsilon - v_0)dx, \\ \mathcal{E}_6(\varepsilon) &= \int_{\omega_\varepsilon} [(F_1 - F_0)(x_0)v_0(x_0) - (F_1 - F_0)v_0]dx. \end{aligned}$$

Again it will be proved that $|\mathcal{E}_i(\varepsilon)| = o(\varepsilon^d)$ for $i = 5, 6$. Consequently we set

$$\delta \ell = |\omega|(F_1 - F_0)(x_0)v_0(x_0).$$

6 Variation of the cost function

On account of Proposition 2.1 and the previous statements, the main result of this paper reads as follows.

Theorem 6.1. *For a cost function satisfying (2.7), (2.8) and (3.7) with $f(\varepsilon) = \varepsilon^d$, we have the asymptotic expansion*

$$j(\varepsilon) - j(0) = \varepsilon^d \left[\alpha_0 \nabla u_0(x_0)^T \mathcal{P}_{\omega,r} \nabla v_0(x_0) + (\beta_1 - \beta_0) |\omega| u_0(x_0) v_0(x_0) - |\omega| (F_1 - F_0)(x_0) v_0(x_0) + \delta J \right] + o(\varepsilon^d). \quad (6.1)$$

The notations

$$\delta J = \delta J_1 + \delta J_2, \quad r = \frac{\alpha_1}{\alpha_0} \geq 0$$

are used for convenience. For $r = 1$, the polarization matrix $\mathcal{P}_{\omega,r}$ is zero. Otherwise, it has the entries

$$(\mathcal{P}_{\omega,r})_{ij} = \int_{\partial\omega} p_i x_j ds \quad (6.2)$$

where x_j is the j^{th} coordinate of the point x and the density p_i associated to the i^{th} basis vector e_i of \mathbb{R}^d is the unique solution of the integral equation

$$\frac{r+1}{r-1} \frac{p_i(x)}{2} + \int_{\partial\omega} p_i(y) A \nabla E(x-y) \cdot n(x) ds(y) = A e_i \cdot n(x) \quad \forall x \in \partial\omega. \quad (6.3)$$

The notion of polarization matrix has been introduced by Polya, Schiffer and Szegö [19, 21], and since then it has been extensively studied (see *e.g.* [6, 7] and the references therein). In particular, it is proved that $\mathcal{P}_{\omega,r}$ is symmetric positive definite if $r > 1$, and symmetric negative definite if $r < 1$. It can be determined analytically in some cases, otherwise it can be approximated numerically. We refer *e.g.* to [17, 15] for some examples. For completeness, let us recall the results obtained for ellipses and ellipsoids in the case of the Laplacian. Note that [15] addresses the anisotropic case where α_0 and α_1 are matrices but, while it can be readily shown that the polarization matrix constructed in [15] coincides with $\mathcal{P}_{\omega,r}$ when $\gamma := \alpha_1 \alpha_0^{-1} = rI$, Formula (6.1) is only valid in the case where α_0 and α_1 are scalars.

1. *Ellipse.* For ω being an ellipse whose axes with semi-length a and $b = ea$ are parallel to the main axes of the coordinate system (the general case can be obtained by a rotation), the polarization matrix reads

$$\mathcal{P}_{\omega,r} = |\omega| (r-1) \begin{pmatrix} \frac{1+e}{1+re} & 0 \\ 0 & \frac{1+e}{e+r} \end{pmatrix}. \quad (6.4)$$

In particular we have for the unit disc

$$\mathcal{P}_{\omega,r} = 2 \frac{r-1}{r+1} |\omega| I. \quad (6.5)$$

2. *Ellipsoid.* The polarization tensor associated to an ellipsoid with semi-axes $(a_i)_{i \in \{1,2,3\}}$, oriented along the main axes of the coordinate system reads

$$\mathcal{P}_{\omega,r} = |\omega| \begin{pmatrix} \frac{r-1}{1-(r-1)s_1} & 0 & 0 \\ 0 & \frac{r-1}{1-(r-1)s_2} & 0 \\ 0 & 0 & \frac{r-1}{1-(r-1)s_3} \end{pmatrix} \quad (6.6)$$

with

$$s_k = -\frac{a_1 a_2 a_3}{2} \int_0^\infty \frac{1}{(a_k^2 + s) \sqrt{(a_1^2 + s)(a_2^2 + s)(a_3^2 + s)}} ds.$$

Using Maple, we obtain for an ellipsoid of revolution with radius $a_1 = a_2$ and height $a_3 = ea_1$, $e < 1$

$$s_1 = s_2 = e \frac{2e\sqrt{1-e^2} + 2 \arctan\left(\frac{e}{\sqrt{1-e^2}}\right) - \pi}{4(1-e^2)^{3/2}}, \quad (6.7)$$

$$s_3 = -e \frac{2\sqrt{1-e^2} + 2 \arctan\left(\frac{e}{\sqrt{1-e^2}}\right) - \pi}{2(1-e^2)^{3/2}}, \quad (6.8)$$

and for a sphere

$$\mathcal{P}_{\omega,r} = 3 \frac{r-1}{r+2} |\omega| I. \quad (6.9)$$

It is interesting to notice that, in 2D, the polarization matrix has 3 degrees of freedom (due to the symmetry), which is equal to the number of degrees of freedom of an ellipse. Hence all inclusions are equivalent (in the sense of the first order topological sensitivity) to an ellipse. The same observation holds in 3D for ellipsoids.

Another remark concerning ellipses and ellipsoids is that they enable to formally retrieve the formulas associated to the creation of a crack. Indeed, taking $r = 0$ (Neumann condition) and $e \rightarrow 0$ in (6.4), (6.7), (6.8) and (6.6) provides

$$\mathcal{P}_{\Sigma,0} = -\pi nn^T \quad \text{for a linear crack } \Sigma \text{ of length 2 and unit normal } n \text{ (2D),}$$

$$\mathcal{P}_{\Sigma,0} = -\frac{8}{3} nn^T \quad \text{for a planar crack } \Sigma \text{ of radius 1 and unit normal } n \text{ (3D),}$$

which coincide with the polarization matrices derived rigorously in [8] with the help of a double layer potential.

7 Particular cost functions

The contribution δJ is explicated here for some examples of cost function.

Theorem 7.1. *The asymptotic expansion (6.1) holds true for the following cost functions with the indicated values of δJ .*

1. **First example.** *We consider a functional of the form*

$$J_\varepsilon(u) = J(u|_{\Omega \setminus \overline{B(x_0, R)}}), \quad (7.1)$$

where R is a fixed positive radius and J is \mathcal{C}^2 -Fréchet-differentiable on $H^1(\Omega \setminus \overline{B(x_0, R)})$. Then

$$\delta J = 0.$$

2. **Second example.** *For the functional*

$$J_\varepsilon(u) = \int_{\Omega} \alpha_\varepsilon |u - u_d|^2 dx, \quad (7.2)$$

with $u_d \in H^2(\Omega)$, we have

$$\delta J = (\alpha_1 - \alpha_0) |\omega| |u_0(x_0) - u_d(x_0)|^2.$$

3. **Third example.** *For the functional*

$$J_\varepsilon(u) = \int_{\Omega} \alpha_\varepsilon A \nabla(u - u_d) \cdot \nabla(u - u_d) dx, \quad (7.3)$$

with $u_d \in (H_0^1 \cap H^3)(\Omega)$, we have

$$\delta J = \alpha_0 \nabla u_0(x_0)^T \mathcal{P}_{\omega,r} (\nabla u_0(x_0) - \nabla u_d(x_0)) - (\alpha_1 - \alpha_0) |\omega| A \nabla u_d(x_0) \cdot (\nabla u_0(x_0) - \nabla u_d(x_0)),$$

where $\mathcal{P}_{\omega,r}$ is the polarization matrix.

8 Case of a PDE system

Theorems 6.1 and 7.1 can be straightforwardly generalized to the vector case where $u_\varepsilon \in \mathbb{R}^m$, $m \in \mathbb{N}^*$. The changes are the following.

- In every formula, two vectors are multiplied in the sense of the dot product of \mathbb{R}^m .
- The polarization $\mathcal{P}_{\omega,r}$ is given by a tensor of order 4. Denoting by \mathcal{P}_{pq}^{ij} its components, we have by definition

$$\nabla u^T \mathcal{P}_{\omega,r} \nabla v = \sum_{i,j,p,q} \mathcal{P}_{pq}^{ij} \partial_i u_j \partial_p v_q.$$

In this framework, the linear elasticity system is of particular interest for the applications. The associated polarization tensor, also called elastic moment tensor (EMT), is studied *e.g.* in [7]. This book provides notably its expression in 2D for an ellipse. Using standard notations in elasticity, it reads for instance for a disc in plane strain

$$\nabla u^T \mathcal{P}_{\omega,r} \nabla v = \frac{r-1}{\kappa r+1} \frac{\kappa+1}{2} |\omega| \left[2\sigma(u) : e(v) + \frac{(r-1)(\kappa-2)}{\kappa+2r-1} \text{tr}\sigma(u) \text{tr}e(v) \right],$$

where u and v stand for any displacement fields, $\sigma(u)$ and $e(v)$ are the corresponding stress and strain tensors and

$$\kappa = \frac{\lambda + 3\mu}{\lambda + \mu}$$

with the Lamé coefficients λ and μ . In plane stress, $\lambda^* = 2\mu\lambda/(\lambda + 2\mu)$ must be substituted for λ . When $r = 0$ (hole with Neumann boundary condition), we retrieve the topological sensitivity obtained in [13].

For a complex solution, namely $u_\varepsilon \in \mathbb{C}^m$, then the hermitian dot product of this space is involved. In addition, since the cost function remains real-valued, a real part appears in front of its asymptotic expansion (again, the reader is referred to [20, 8] for details). Formula (6.1) becomes

$$j(\varepsilon) - j(0) = \varepsilon^d \left[\Re[\alpha_0(\mathcal{P}_{\omega,r} \nabla u_0(x_0)) \cdot \nabla v_0(x_0)] + (\beta_1 - \beta_0) |\omega| u_0(x_0) \cdot v_0(x_0) - |\omega| (F_1 - F_0)(x_0) \cdot v_0(x_0) + \delta J \right] + o(\varepsilon^d), \quad (8.1)$$

where the dot stands for the hermitian dot product of \mathbb{C}^m .

9 Proofs

In this section the letter c stands for any positive constant that may change from place to place but is always independent of ε . Possibly shifting the origin of the coordinate system, we suppose for simplicity that $x_0 = 0$. We denote by R some fixed radius such that $\overline{B(0, R)} \subset \Omega$. For the sake of readability, the duality pairings between H^{-1} and H^1 and between $H^{-1/2}$ and $H^{1/2}$ are denoted by integrals.

9.1 Preliminary estimates

Lemma 9.1. *Let h_ε be the function defined by (4.5) and (4.6). We have*

$$\|h_\varepsilon\|_{0,\Omega} = o(\varepsilon^{d/2}), \quad (9.1)$$

$$|h_\varepsilon|_{1,\Omega} = O(\varepsilon^{d/2}), \quad (9.2)$$

$$|h_\varepsilon|_{1,\Omega \setminus \overline{B(0,R)}} = O(\varepsilon^d). \quad (9.3)$$

Proof. According to the integral representation (4.8) together with the condition (4.7), we have that

$$H(x) = \int_{\partial\omega} \frac{p(y)}{\alpha_1 - \alpha_0} [E(x-y) - E(x)] ds(y).$$

It comes

$$\begin{aligned} |H(x)| &\leq c \sup_{y \in \partial\omega} |E(x-y) - E(x)| \\ &\leq c \sup_{|z| \leq R_\omega} |\nabla E(x-z)|, \end{aligned}$$

where R_ω denotes a radius such that $\omega \subset B(0, R_\omega)$. When $|x|$ tends to infinity, $|\nabla E(x)| = O(|x|^{1-d})$. We deduce that $|H(x)| = O(|x|^{1-d})$ as well. Similarly we get $|\nabla H(x)| = O(|x|^{-d})$. It follows the estimates

$$\begin{aligned} \|H\|_{0, \Omega/\varepsilon} &= \begin{cases} O(\sqrt{-\ln \varepsilon}) & \text{in 2D,} \\ O(1) & \text{in 3D,} \end{cases} \\ |H|_{1, \Omega/\varepsilon} &= O(1), \\ |H|_{1, (\Omega \setminus \overline{B(0, R)})/\varepsilon} &= O(\varepsilon^{d/2}). \end{aligned}$$

A change of variable completes the proof. \square

Lemma 9.2. *We have*

$$\|v_\varepsilon - v_0 - h_\varepsilon\|_{1, \Omega} = o(\varepsilon^{d/2}). \quad (9.4)$$

Proof. Let us define the function

$$e_\varepsilon = v_\varepsilon - v_0 - h_\varepsilon$$

and the distribution

$$\Phi_\varepsilon = -\operatorname{div}(\alpha_\varepsilon A \nabla e_\varepsilon) + \beta_\varepsilon e_\varepsilon \in H^{-1}(\Omega).$$

We have for any test function $\varphi \in H_0^1(\Omega)$

$$\begin{aligned} \int_\Omega \Phi_\varepsilon \varphi dx &= \int_\Omega [\alpha_\varepsilon A \nabla e_\varepsilon \cdot \nabla \varphi + \beta_\varepsilon e_\varepsilon \varphi] dx \\ &= \int_\Omega [\alpha_\varepsilon A \nabla v_\varepsilon \cdot \nabla \varphi + \beta_\varepsilon v_\varepsilon \varphi] dx - \int_\Omega [\alpha_0 A \nabla v_0 \cdot \nabla \varphi + \beta_0 v_0 \varphi] dx \\ &\quad - \int_{\omega_\varepsilon} [(\alpha_1 - \alpha_0) A \nabla v_0 \cdot \nabla \varphi + (\beta_1 - \beta_0) v_0 \varphi] dx \\ &\quad - \int_\Omega [\alpha_\varepsilon A \nabla h_\varepsilon \cdot \nabla \varphi + \beta_\varepsilon h_\varepsilon \varphi] dx. \end{aligned}$$

Then, evaluating the first two terms thanks to the adjoint equation, the third one by an integration by parts and the fourth one by using the definition of h_ε yields

$$\begin{aligned} \int_\Omega \Phi_\varepsilon \varphi dx &= -((DJ_\varepsilon(u_0) - DJ_0(u_0))\varphi \\ &\quad + \int_{\omega_\varepsilon} [(\alpha_1 - \alpha_0) \operatorname{div}(A \nabla v_0)(\varphi - \bar{\varphi}) - (\beta_1 - \beta_0) v_0 \varphi] dx \\ &\quad - (\alpha_1 - \alpha_0) \int_{\partial\omega_\varepsilon} (A \nabla v_0 \cdot n)(\varphi - \bar{\varphi}) ds \\ &\quad + (\alpha_1 - \alpha_0) \int_{\partial\omega_\varepsilon} (A \nabla v_0(0) \cdot n) \varphi ds - \int_\Omega \beta_\varepsilon h_\varepsilon \varphi dx. \end{aligned}$$

The constant

$$\bar{\varphi} = \frac{1}{|\omega_\varepsilon|} \int_{\omega_\varepsilon} \varphi dx = \frac{1}{|\omega|} \int_\omega \varphi(\varepsilon x) dx$$

is introduced for convenience. A rearrangement and a change of variable lead to

$$\begin{aligned} \int_{\Omega} \Phi_{\varepsilon} \varphi dx &= -((DJ_{\varepsilon}(u_0) - DJ_0(u_0))\varphi \\ &+ \int_{\omega_{\varepsilon}} [(\alpha_1 - \alpha_0)\operatorname{div}(A\nabla v_0)(\varphi - \bar{\varphi}) - (\beta_1 - \beta_0)v_0\varphi] dx \\ &- \varepsilon^{d-1}(\alpha_1 - \alpha_0) \int_{\partial\omega} A[\nabla v_0(\varepsilon x) - \nabla v_0(0)].n[\varphi(\varepsilon x) - \bar{\varphi}] ds - \int_{\Omega} \beta_{\varepsilon} h_{\varepsilon} \varphi dx. \end{aligned}$$

Thanks to the hypotheses, $v_0 \in H^4(\Omega) \subset C^2(\bar{\Omega})$. Using furthermore the Hölder inequality with some coefficients p and q satisfying $1/p + 1/q = 1$, it comes

$$\begin{aligned} \left| \int_{\Omega} \Phi_{\varepsilon} \varphi dx \right| &\leq \|DJ_{\varepsilon}(u_0) - DJ_0(u_0)\|_{-1,\Omega} \|\varphi\|_{1,\Omega} \\ &+ c\|1\|_{L^p(\omega_{\varepsilon})} \|\varphi\|_{L^q(\omega_{\varepsilon})} + c\varepsilon^d |\bar{\varphi}| \\ &+ c\varepsilon^d \|\varphi(\varepsilon x) - \bar{\varphi}\|_{1/2,\partial\omega} + c\|h_{\varepsilon}\|_{0,\Omega} \|\varphi\|_{0,\Omega}. \end{aligned}$$

The first term is bounded by the assumption (3.7). The Sobolev imbedding theorem provides $\|\varphi\|_{L^q(\omega_{\varepsilon})} \leq c\|\varphi\|_{1,\Omega}$ for all $q < +\infty$ in 2D, $q \leq 6$ in 3D. Let us choose such a q greater than 2 and take the corresponding $p = q/(q-1)$. Another application of the Hölder inequality with the same coefficients p and q furnishes

$$|\bar{\varphi}| \leq c\varepsilon^{d/p-d} \|\varphi\|_{L^q(\omega_{\varepsilon})}.$$

Besides, the trace theorem, the equivalence of the norm and the semi-norm on the subspace of $H^1(\omega)$ of the functions with zero mean value and a change of variable bring

$$\|\varphi(\varepsilon x) - \bar{\varphi}\|_{1/2,\partial\omega} \leq c\|\varphi(\varepsilon x) - \bar{\varphi}\|_{1,\omega} \leq c|\varphi(\varepsilon x)|_{1,\omega} = c\varepsilon^{1-d/2} |\varphi|_{1,\omega_{\varepsilon}}.$$

Hence, using Lemma 9.1 to estimate the last term, we get

$$\left| \int_{\Omega} \Phi_{\varepsilon} \varphi dx \right| = o(\varepsilon^{d/2}) \|\varphi\|_{1,\Omega} + O(\varepsilon^{d/p}) \|\varphi\|_{1,\Omega} + O(\varepsilon^{d/2+1}) |\varphi|_{1,\omega_{\varepsilon}} + o(\varepsilon^{d/2}) \|\varphi\|_{0,\Omega}.$$

As $1/p > 1/2$, it follows that

$$\|\Phi_{\varepsilon}\|_{-1,\Omega} = o(\varepsilon^{d/2}). \quad (9.5)$$

Let us now come back to the study of the function e_{ε} . The two cases have to be treated separately.

1. *First case* ($\alpha_1 > 0$). We have

$$\begin{cases} -\operatorname{div}(\alpha_{\varepsilon} A\nabla e_{\varepsilon}) + \beta_{\varepsilon} e_{\varepsilon} &= \Phi_{\varepsilon} & \text{in } \Omega, \\ e_{\varepsilon} &= -h_{\varepsilon} & \text{on } \partial\Omega, \end{cases} \quad (9.6)$$

from which we deduce by uniform elliptic regularity (see [24])

$$\|e_{\varepsilon}\|_{1,\Omega} \leq c\|\Phi_{\varepsilon}\|_{-1,\Omega} + c\|h_{\varepsilon}\|_{1/2,\partial\Omega}.$$

Thanks to (9.5) and Lemma 9.1 it comes the desired estimate.

2. *Second case* ($\alpha_1 = \beta_1 = 0$). In order to get unique solvability and elliptic regularity we must restrict (9.6) to the perforated domain $\Omega_{\varepsilon} := \Omega \setminus \bar{\omega}_{\varepsilon}$:

$$\begin{cases} -\operatorname{div}(\alpha_0 A\nabla e_{\varepsilon}) + \beta_0 e_{\varepsilon} &= \Phi_{\varepsilon} & \text{in } \Omega_{\varepsilon}, \\ e_{\varepsilon} &= -h_{\varepsilon} & \text{on } \partial\Omega, \\ A e_{\varepsilon}.n &= -A v_0.n + A v_0(0).n & \text{on } \partial\omega_{\varepsilon}. \end{cases} \quad (9.7)$$

According to an estimate proved in [8] we have

$$\|e_{\varepsilon}\|_{1,\Omega_{\varepsilon}} \leq c\|\Phi_{\varepsilon}\|_{-1,\Omega_{\varepsilon}} + c\|h_{\varepsilon}\|_{1/2,\partial\Omega} + c\varepsilon^{d/2} \|A v_0(\varepsilon x).n + A v_0(0).n\|_{1/2,\partial\omega}.$$

The same arguments as in the previous case together with the regularity of v_0 near the origin yield

$$\|e_\varepsilon\|_{1,\Omega_\varepsilon} = o(\varepsilon^{d/2}). \quad (9.8)$$

Inside the hole, we have $-\operatorname{div}(A\nabla e_\varepsilon) = 0$. Therefore, the standard elliptic regularity and the trace theorem applied to the transported function $e_\varepsilon(\varepsilon x)$ lead to

$$\|e_\varepsilon(\varepsilon x) + \lambda\|_{1,\omega} \leq c\|e_\varepsilon(\varepsilon x) + \lambda\|_{1/2,\partial\omega} \leq c\|e_\varepsilon(\varepsilon x) + \lambda\|_{1,B \setminus \bar{\omega}}$$

for every $\lambda \in \mathbb{R}$ and any fixed open set B containing $\bar{\omega}$. Due to the equivalence of the norm and semi-norm in the quotient spaces $H^1(\omega)/\mathbb{R}$ and $H^1(B \setminus \bar{\omega})/\mathbb{R}$ it comes

$$|e_\varepsilon(\varepsilon x)|_{1,\omega} \leq c|e_\varepsilon(\varepsilon x)|_{1,B \setminus \bar{\omega}}.$$

A change of variable brings

$$|e_\varepsilon|_{1,\omega_\varepsilon} \leq c|e_\varepsilon|_{1,\Omega_\varepsilon}. \quad (9.9)$$

Thanks to the Poincaré inequality applied to the function $e_\varepsilon - Rh_\varepsilon$ where Rh_ε denotes a lifting of the trace of h_ε on $\partial\Omega$ whose support is contained in Ω deprived of a neighborhood of the origin, we obtain that

$$\|e_\varepsilon\|_{1,\Omega} \leq \|e_\varepsilon - Rh_\varepsilon\|_{1,\Omega} + \|Rh_\varepsilon\|_{1,\Omega} \leq c|e_\varepsilon|_{1,\Omega} + c\|h_\varepsilon\|_{1/2,\partial\Omega}.$$

The proof is completed by the application of Lemma 9.1 and the estimates (9.9) and (9.8). □

Lemma 9.3. *We have*

$$\|v_\varepsilon - v_0\|_{0,\Omega} = o(\varepsilon^{d/2}), \quad (9.10)$$

$$\|v_\varepsilon - v_0\|_{1,\Omega} = O(\varepsilon^{d/2}), \quad (9.11)$$

$$\|v_\varepsilon - v_0\|_{1,\Omega \setminus \overline{B(0,R)}} = o(\varepsilon^{d/2}). \quad (9.12)$$

The same estimates hold for the direct state, i.e. by substituting u_ε and u_0 for v_ε and v_0 .

Proof. It is a combination of Lemmas 9.1 and 9.2. Since the direct state solves the same PDE as the adjoint, but with a right hand side whose variation also satisfies $\|F_\varepsilon - F_0\|_{-1,\Omega} = o(\varepsilon^{d/2})$, all the obtained estimates can be extended. □

9.2 Proof of Theorem 6.1

We shall prove here that $\mathcal{E}_i(\varepsilon) = o(\varepsilon^d)$ for $i = 1, \dots, 6$.

1. We obtain straightforwardly from the regularity of u_0 by using *e.g.* the Taylor-Lagrange inequality the bound

$$|\mathcal{E}_1(\varepsilon)| \leq c\varepsilon^{d+1}.$$

2. Using again the regularity of u_0 we can write that

$$|\mathcal{E}_2(\varepsilon)| \leq c \int_{\omega_\varepsilon} |v_\varepsilon - v_0| \, dx.$$

From the Hölder inequality, we obtain that for all $p, q \in [1, +\infty]$ satisfying $1/p + 1/q = 1$,

$$|\mathcal{E}_2(\varepsilon)| \leq c\varepsilon^{d/p} \|v_\varepsilon - v_0\|_{L^q(\omega_\varepsilon)}.$$

Again, we choose $q \in]2, +\infty[$ in 2D, $q \in]2, 6]$ in 3D and p accordingly, so that the Sobolev imbedding theorem provides $H^1(\Omega) \subset L^q(\Omega)$ with a continuous imbedding. Therefore we have

$$|\mathcal{E}_2(\varepsilon)| \leq c\varepsilon^{d/p} \|v_\varepsilon - v_0\|_{1,\Omega}.$$

Then Lemma 9.3 implies

$$|\mathcal{E}_2(\varepsilon)| \leq c\varepsilon^{kd}$$

with $k = \frac{1}{2} + \frac{1}{p} = \frac{3}{2} - \frac{1}{q} > 1$.

3. We have

$$\begin{aligned} |\mathcal{E}_3(\varepsilon)| &\leq c|u_0|_{1,\omega_\varepsilon}|v_\varepsilon - v_0 - h_\varepsilon|_{1,\omega_\varepsilon} \\ &\leq c\varepsilon^{d/2}\|u_0\|_{C^1(\omega_\varepsilon)}\|v_\varepsilon - v_0 - h_\varepsilon\|_{1,\Omega}. \end{aligned}$$

Then Lemma 9.2 yields

$$|\mathcal{E}_3(\varepsilon)| = o(\varepsilon^d).$$

4. It comes again straightforwardly from the Taylor-Lagrange inequality that

$$|\mathcal{E}_4(\varepsilon)| \leq c\varepsilon^{d+1}.$$

5. In the same way as we estimated $\mathcal{E}_2(\varepsilon)$ we obtain

$$|\mathcal{E}_5(\varepsilon)| = o(\varepsilon^d).$$

6. The regularity of F_1 , F_0 and v_0 in the vicinity of the origin yields immediately

$$|\mathcal{E}_6(\varepsilon)| \leq c\varepsilon^{d+1},$$

which ends up the proof. \square

9.3 Proof of Theorem 7.1

We shall check that Equations (2.7), (2.8) and (3.7) hold for the proposed examples.

1. In the first example, the functional is independent of ε , so that (2.8) and (3.7) are automatically satisfied. As J is of class \mathcal{C}^2 , we have

$$J_\varepsilon(u_\varepsilon) - J_\varepsilon(u_0) - DJ_\varepsilon(u_0)(u_\varepsilon - u_0) = O(\|u_\varepsilon - u_0\|_{1,\Omega \setminus \overline{B(0,R)}}^2)$$

which, due to Lemma 9.3, leads to (2.7).

2. Let us now consider the second example.

Expression of $J_\varepsilon(u_\varepsilon) - J_\varepsilon(u_0)$. We have

$$J_\varepsilon(u_\varepsilon) - J_\varepsilon(u_0) = 2 \int_{\Omega} \alpha_\varepsilon(u_\varepsilon - u_0)(u_0 - u_d) dx + \int_{\Omega} \alpha_\varepsilon |u_\varepsilon - u_0|^2 dx.$$

Using Lemma 9.3 we obtain that

$$J_\varepsilon(u_\varepsilon) - J_\varepsilon(u_0) = DJ_\varepsilon(u_0)(u_\varepsilon - u_0) + o(\varepsilon^d),$$

with

$$DJ_\varepsilon(u_0)(\varphi) = 2 \int_{\Omega} \alpha_\varepsilon \varphi (u_0 - u_d) dx \quad \forall \varphi \in H^1(\Omega).$$

Hence

$$\delta J_1 = 0.$$

Expression of $J_\varepsilon(u_0) - J_0(u_0)$. We have

$$\begin{aligned} J_\varepsilon(u_0) - J_0(u_0) &= \int_{\Omega} \alpha_\varepsilon |u_0 - u_d|^2 dx - \alpha_0 \int_{\Omega} |u_0 - u_d|^2 dx \\ &= (\alpha_1 - \alpha_0) \int_{\omega_\varepsilon} |u_0 - u_d|^2 dx. \end{aligned}$$

Thanks to the regularity of u_0 and u_d we obtain easily that

$$J_\varepsilon(u_0) - J_0(u_0) = (\alpha_1 - \alpha_0)|\omega|\varepsilon^d|u_0(0) - u_d(0)|^2 + o(\varepsilon^d).$$

Thus

$$\delta J_2 = (\alpha_1 - \alpha_0)|\omega||u_0(0) - u_d(0)|^2.$$

Estimation of $\|DJ_\varepsilon(u_0) - DJ_0(u_0)\|_{-1,\Omega}$. We have for all $\varphi \in H_0^1(\Omega)$

$$(DJ_\varepsilon(u_0) - DJ_0(u_0))\varphi = 2(\alpha_1 - \alpha_0) \int_{\omega_\varepsilon} \varphi(u_0 - u_d) dx.$$

The Hölder inequality, the Sobolev imbedding theorem and the fact that u_0 and u_d are continuous in a neighborhood of the origin yield successively, for ε sufficiently small, $1/p + 1/q = 1$ and $q < +\infty$ in 2D, $q \leq 6$ in 3D,

$$\begin{aligned} |(DJ_\varepsilon(u_0) - DJ_0(u_0))\varphi| &\leq c\|u_0 - u_d\|_{L^p(\omega_\varepsilon)}\|\varphi\|_{L^q(\omega_\varepsilon)} \\ &\leq c\varepsilon^{d/p}\|\varphi\|_{1,\Omega}. \end{aligned}$$

Noticing again that $1/p$ can be chosen strictly greater than $1/2$ proves that Equation (3.7) holds.

3. It can be easily seen that the third cost functional does not satisfy (2.7). However, this difficulty can be overcome by constructing an auxiliary functional that coincides with (7.3) when it is evaluated at the direct state u_ε but whose general expression meets the requirements. Indeed, the Green formula leads to the identity

$$J_\varepsilon(u_\varepsilon) = \tilde{J}_\varepsilon(u_\varepsilon),$$

where the functional $\tilde{J}_\varepsilon(u_\varepsilon)$ is defined by

$$\tilde{J}_\varepsilon(u) = \int_{\Omega} F_\varepsilon(u - u_d) dx - \int_{\Omega} \beta_\varepsilon u(u - u_d) dx - \int_{\Omega} \alpha_\varepsilon A \nabla u_d \cdot \nabla(u - u_d) dx.$$

The second term can be studied in the same way as functional (7.2). The two other terms are constant or linear with respect to u , so they can be treated easily. We arrive at the conclusion that \tilde{J}_ε satisfies (2.7), (2.8) and (3.7) with

$$\delta \tilde{J}_1 = 0,$$

$$\begin{aligned} \delta \tilde{J}_2 = & |\omega|[(F_1 - F_0)(0)(u_0 - u_d)(0) - (\beta_1 - \beta_0)u_0(0)(u_0(0) - u_d(0)) \\ & - (\alpha_1 - \alpha_0)A \nabla u_d(0) \cdot (\nabla u_0(0) - \nabla u_d(0))]. \end{aligned}$$

However, we must take care to the fact that the adjoint state \tilde{v}_ε associated to the functional \tilde{J}_ε differs from the adjoint v_ε associated to J_ε . By comparing both adjoint equations we derive the relation

$$\tilde{v}_\varepsilon = v_\varepsilon + 2u_0 - u_d - u_\varepsilon.$$

Then the asymptotic expansion of the criterion $j(\varepsilon) = J_\varepsilon(u_\varepsilon) = \tilde{J}_\varepsilon(u_\varepsilon)$ is given by Equation (6.1) with $\tilde{v}_0 = v_0 + u_0 - u_d$ substituted for v_0 and $\delta \tilde{J} = \delta \tilde{J}_1 + \delta \tilde{J}_2$ substituted for δJ . A rearrangement yields

$$\begin{aligned} j(\varepsilon) - j(0) = & \varepsilon^d[\alpha_0 \nabla u_0(0)^T \mathcal{P}_{\omega,r} \nabla v_0(0) + (\beta_1 - \beta_0)|\omega|u_0(0)v_0(0) \\ & - |\omega|(F_1 - F_0)(0)v_0(0) + \delta J] + o(\varepsilon^d) \quad (9.13) \end{aligned}$$

with the announced value of δJ . \square

10 A numerical application

The sensitivity formula obtained before is used to identify dielectric objects with the help of electromagnetic waves and boundary measurements in 2D. Consider an open and bounded subset Ω of \mathbb{R}^2 whose boundary Γ is a regular polygon of N sides $\Gamma_i, i = 1, \dots, N$. A dielectric

object whose properties are known may lie inside Ω . On each side of the external boundary Γ is successively emitted an electromagnetic wave. The problem is modeled as follows:

$$\begin{cases} -\operatorname{div}(\alpha \nabla u_i^m) + \beta u_i^m = 0 & \text{in } \Omega, \\ \partial_n u_i^m - \mathbf{i}k u_i^m = 0 & \text{on } \Gamma_l, l \neq i, \\ \partial_n u_i^m - \mathbf{i}k u_i^m = -2\mathbf{i}k & \text{on } \Gamma_i, \end{cases}$$

with $\mathbf{i}^2 = -1$. In this system, u_i^m stands for the vertical component of the electric field for an H -plane polarization (*i.e.* the magnetic field vector is contained in the horizontal plane where the wave propagates and the electric field is vertical), whereas it stands for the vertical component of the magnetic field for an E -plane polarization (conversely, the electric field is horizontal and the electric field vertical). The coefficients α and β are piecewise constant functions of the point x , respectively equal to α_1 and β_1 inside \mathcal{O} (see Table 1), $\alpha_0 = 1$, $\beta_0 = -k^2$ outside. The letter k denotes the wave number.

We assume that we have at our disposal the measurements

$$S_{ij}^m = S_j(u_i^m) = \int_{\Gamma_j} u_i^m ds, \quad i, j = 1, \dots, N.$$

To detect the actual object thanks to the knowledge of the matrix $(S_{ij}^m)_{i,j=1,\dots,N}$, we look for the best locations where to insert small inhomogeneities in order to minimize the cost function

$$J(u_1, \dots, u_N) = \sum_{i=1}^N \sum_{j=1}^N |S_j(u_i) - S_{ij}^m|^2.$$

For a circular inhomogeneity, according to (8.1), that information is provided by the sensitivity:

$$g(x) = \sum_{i=1}^N \Re \left(\frac{2\alpha_0(\alpha_1 - \alpha_0)}{\alpha_0 + \alpha_1} \nabla u_i(x) \cdot \nabla v_i(x) + (\beta_1 - \beta_0) u_i(x) \cdot v_i(x) \right),$$

where the N direct states u_i and the N adjoint states v_i are defined by:

$$\begin{cases} \Delta u_i + k^2 u_i = 0 & \text{in } \Omega, \\ \partial_n u_i - \mathbf{i}k u_i = 0 & \text{on } \Gamma_l, l \neq i, \\ \partial_n u_i - \mathbf{i}k u_i = -2\mathbf{i}k & \text{on } \Gamma_i, \end{cases}$$

$$\begin{cases} \Delta \bar{v}_i + k^2 \bar{v}_i = 0 & \text{in } \Omega, \\ \partial_n \bar{v}_i - \mathbf{i}k \bar{v}_i = -2 \overline{(S_j(u_i) - S_{ij}^m)} & \text{on } \Gamma_j, \end{cases}$$

the bar denoting the complex conjugacy.

Figure 1 represents the topological gradient computed in three different configurations (corresponding to different values of the index of refraction of the sought object) with the parameters $k = 10$ and $N = 32$. The indicated values of μ_r and ν correspond to an E -plane polarization. The measurements are simulated numerically. In one iteration, the location of the object is clearly pointed out by the negative peak of the topological gradient whereas the observation of the isovalues gives a rough idea of its shape.

References

- [1] C. Alves, H. Ammari. *Boundary integral formulas for the reconstruction of imperfections of small diameter in an elastic medium*, SIAM J. Appl. Math. 62 (2001), 94-106.
- [2] H. Ammari. *An inverse initial boundary value problem for the wave equation in the presence of imperfections of small volume*, SIAM J. Control Optim. 41 (2003), 1194-1211.
- [3] H. Ammari, H. Kang. *A new method for reconstructing electromagnetic inhomogeneities of small volume*, Inverse problems 19 (2003), 63-71.
- [4] H. Ammari, D. Volkov. *Asymptotic formulas for perturbations in the eigenfrequencies of the full Maxwell equations due to the presence of imperfections of small diameter*, Asympt. Anal. 30 (2002), 331-350.
- [5] H. Ammari, S. Moskow. *Asymptotic expansions for eigenvalues in the presence of small inhomogeneities*, Math. Meth. Appl. Sci. 26 (2003), 67-75.
- [6] H. Ammari, H. Kang, K. Kim. *Polarization tensors and effective properties of anisotropic composite materials*, J. Differ. Equat 215 (2005), 401-428.
- [7] H. Ammari, H. Kang. *Reconstruction of small inhomogeneities from boundary measurements*, Lecture Notes in Mathematics 1846, Springer, 2004.
- [8] S. Amstutz, N. Dominguez. *Topological sensitivity analysis in the context of ultrasonic nondestructive testing*, RICAM-Report No. 2005-21.
- [9] M. Brühl, M. Hanke, M.S. Vogelius. *A direct impedance tomography algorithm for locating small inhomogeneities*, Numer. Math. 93 (2003), 635-654.
- [10] J. Céa, S. Garreau, Ph. Guillaume, M. Masmoudi, *The shape and topological optimizations connections*, Comput. Methods Appl. Mech. Engrg. 188 (2000), 713-726.
- [11] D.J. Cedio-Fengya, S. Moskow, M.S. Vogelius. *Identification of conductivity imperfections of small diameter by boundary measurements. Continuous dependence and computational reconstruction*, Inverse problems 14 (2998), 553-595.
- [12] R. Dautray, J.-L. Lions, *Analyse mathématique et calcul numérique pour les sciences et les techniques*, Masson, collection CEA, 1987.
- [13] S. Garreau, Ph. Guillaume, M. Masmoudi, *The topological Asymptotic for PDE systems: the elasticity case*, SIAM J. Control Optim. 39(6) (2001), 1756-1778.
- [14] P. Guillaume, K. Sididris. *The topological asymptotic expansion for the Dirichlet problem*, SIAM J. Control Optim. 41(4) (2002), 1042-1072.
- [15] H. Kang, K. Kim. *Anisotropic polarization tensors for ellipses and ellipsoids*, submitted.
- [16] M. Masmoudi. *The Topological Asymptotic*, in Computational Methods for Control Applications, R. Glowinski, H. Kawarada and J. Periaux eds, GAKUTO Internat. Ser. Math. Sci. Appl. 16 (2001), 53-72.
- [17] G.W. Milton. *The theory of composites*, Cambridge Monographs on Applied and Computational Mathematics, Cambridge University Press, 2001.
- [18] A.A. Novotny, R.A. Feijóo, E. Taroco, C. Padra. *Topological sensitivity analysis*, Comput. Methods Appl. Mech. Engrg. 192 (2003), 803-829.
- [19] G. Polya, G. Szegő. *Isoperimetric inequalities in mathematical physics*, Annals of Mathematical Studies 27, Princeton University Press, 1951.

- [20] B. Samet, S. Amstutz, M. Masmoudi. *The topological asymptotic for the Helmholtz equation*, SIAM J. Control Optim. 42(5) (2003), 1523-1544.
- [21] M. Schiffer, G. Szegö. *Virtual mass and polarization*, Trans. Amer. Math. Soc. 67 (1949), 130-205.
- [22] A. Schumacher, V.V. Kobolev, H.A. Eschenauer, *Bubble method for topology and shape optimization of structures*, Journal of structural optimization 8 (1994), 42-51.
- [23] J. Sokolowski, A. Żochowski, *On the topological derivative in shape optimization*, SIAM J. Control Optim. 37 (1999), 1241-1272.
- [24] M.S. Vogelius, D. Volkov. *Asymptotic formulas for perturbations in the electromagnetic fields due to the presence of inhomogeneities of small diameter*, Math. Model. Numer. Anal. 34 (2000), 723-748.

	α_1	β_1
<i>H</i> -plane	$\frac{1}{\nu^2}$	$-\frac{\mu_r}{\nu^2 k^2}$
<i>E</i> -plane	$\frac{\mu_r}{\nu^2}$	$-\mu_r k^2$

Table 1: PDE coefficients in electromagnetism (ν and μ_r denote the index of refraction and the relative permeability of the object, respectively, and k is the wave number).

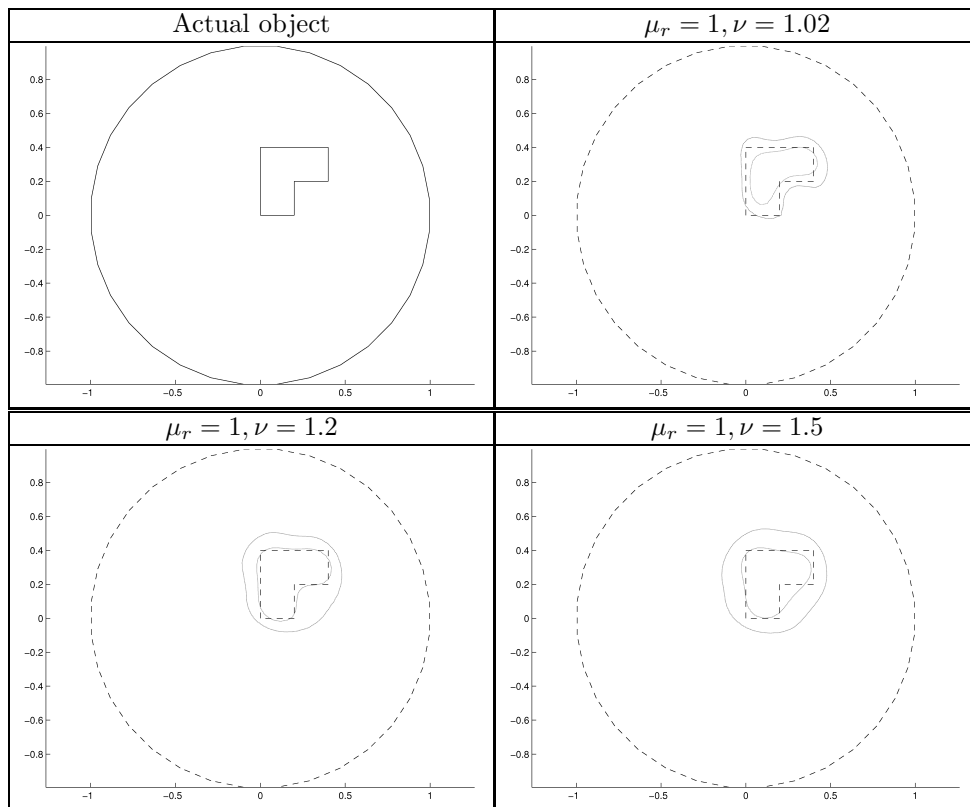


Figure 1: The actual object and two negative isovalues of the topological gradient.