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Topological sensitivity analysis in the context of ultrasonic nondestructive testing

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Abstract

This paper deals with the use of the topological derivative in detection problems involving waves. In the first part, a framework to carry out the topological sensitivity analysis in this context is proposed. Arbitrarily shaped holes and cracks with Neumann boundary condition in 2 and 3 space dimensions are considered. In the second part, a numerical example concerning the treatment of ultrasonic probing data in metallic plates is presented. With moderate noise in the measurements, the defects (air bubbles) are detected and satisfactorily localized by means of a single sensitivity computation.

Key words: topological sensitivity, topological gradient, topological derivative, nondestructive testing.

1 Introduction

Inspection problems can generally be seen as shape inversion problems. If techniques borrowed from shape optimization are now commonly accepted as good theoretical candidates to address shape inversion problems, their applications to inspection problems such as nondestructive testing or medical imaging are today relatively restricted. The main reason is that, in such problems, the possibility to handle topology changes is crucial. Therefore the use

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of the topological derivative concept, which directly deals with the variable "topology", seems to be particularly well-suited. We recall the basic principles of this approach, introduced by Schumacher [24], Sokolowski and Zochowski [25] in structural optimization. Consider a cost function $\mathcal{J}(\Omega) = J_\Omega(u_\Omega)$ where u_Ω is the solution of a system of partial differential equations defined in the domain $\Omega \subset \mathbb{R}^N$, $N = 2$ or 3, a point $x_0 \in \Omega$ and a fixed open and bounded subset ω of $\mathbb{R}^N$ containing the origin. The "topological asymptotic expansion" is an expression of the form

$$\mathcal{J}(\Omega \setminus (\overline{x_0 + \rho\omega})) - \mathcal{J}(\Omega) = f(\rho)g(x_0) + o(f(\rho)), \tag{1}$$

where $f(\rho)$ is a positive function tending to zero with ρ. Therefore, to minimize $\mathcal{J}(\Omega)$, we have interest to remove matter where the "topological gradient" (also called "topological derivative", or "topological sensitivity") g is negative.

A general framework enabling to calculate the topological asymptotic expansion for a large class of shape functionals has been worked out by Masmoudi [18]. It is based on an adaptation of the adjoint method and a domain truncation technique providing an equivalent formulation of the PDE in a fixed function space. Using this framework, Garreau, Guillaume, Masmoudi and Sididris [12,14,15] have obtained the topological asymptotic expansions for several problems associated with linear and homogeneous differential operators. For such operators, but with a different approach, more general shape functionals are considered in [19]. The link between the shape and the topological derivatives has been established by Feijóo *et al* [10,21]. This gives rise to a generic method for deriving the latter. However, it seems rather restricted to circular or spherical holes. For the first time a topological sensitivity analysis for a non-homogeneous operator was performed in [23]. The case of a circular hole with a Dirichlet condition imposed on its boundary was considered. For completeness, we point out that extensive research efforts using related techniques have been done in the context of reconstruction problems from boundary measurements (see *e.g.* [1,2,3,7,28] among others). In contrast to our approach, they do not deal with an explicit cost function. Instead, the sensitivity of the PDE solution u_Ω at the location of the measurements (or its integral against special test functions) is computed. Then the data are interpreted by signal processing methods. In addition, those works are focused on the detection of inhomogeneities, which means that the material density inside the inclusions is nonzero.

The problem of interest in this paper is related to nondestructive testing by means of ultrasounds in the context of elastodynamics. The governing equations at a fixed frequency involve a non-homogeneous differential operator of the form

$$u \mapsto \text{div } (A\nabla u) + k^2 u, \tag{2}$$

where A is a symmetric positive definite tensor. For such a problem, the topological asymptotic expansion is determined in dimensions 2 and 3 with respect to the creation of an arbitrarily shaped hole and an arbitrarily shaped crack on which a Neumann boundary condition is prescribed. For the sake of simplicity, the analysis is presented for the Helmholtz operator ($A = I$), but it applies similarly to any operator of the form (2). We introduce an adjoint method that takes into account the variation of the function space, so that a domain truncation is not needed. This formalism brings several technical simplifications, notably for the study of criteria depending explicitly on Ω, for which the truncation necessitates to transport the cost function in the fixed domain (see [14]). Similar results have been obtained in [6]. However, the analysis is not done there in a rigorous mathematical framework. Furthermore, the crack problem is not addressed and less general cost functionals are considered.

The rest of the present paper is organized as follows. The adjoint method is presented in Section 2. The framework of the study is described in Section 3. The topological asymptotic analysis for a hole and a crack are carried out in Sections 4 and 5, respectively, the intermediate proofs being reported in Section 8. The case of some particular cost functions is examined in Section 6. Section 7 is devoted to numerical experiments that highlight the relevance of the topological sensitivity approach for nondestructive testing applications.

2 An appropriate adjoint method

In this section, the adjoint method is generalized to a class of problems for which the state variable belongs to a function space that depends on the control variable. Let $(\mathcal{V}_\rho)_{\rho\geq 0}$ be a family of Hilbert spaces on the complex field such that

$$\mathcal{V}_0 \subset \mathcal{V}_\rho \qquad \forall \rho \geq 0.$$

For all $\rho \geq 0$, let a_ρ be a sesquilinear and continuous form on $\mathcal{V}_\rho$ and let l_ρ be a semilinear and continuous form on $\mathcal{V}_\rho$. We assume that, for all $\rho \geq 0$, the variational problem

$$\begin{cases} u_\rho \in \mathcal{V}_\rho, \\ a_\rho(u_\rho, v) = l_\rho(v) \qquad \forall v \in \mathcal{V}_\rho \end{cases} \tag{3}$$

admits a unique solution. We make the following assumption.

Hypothesis 1 *For all $\rho \geq 0$, there exist on $\mathcal{V}_\rho$ a sesquilinear and continuous form $\tilde{a}_\rho$ and a semilinear and continuous form $\tilde{l}_\rho$ such that*

$$\tilde{a}_\rho(u_0, v) = \tilde{l}_\rho(v) \qquad \forall v \in \mathcal{V}_\rho. \tag{4}$$

Consider now a criterion $j(\rho) = J_\rho(u_\rho) \in \mathbb{R}$ where, for all $\rho \geq 0$, the function J_ρ is differentiable in the following sense: there exists a linear and continuous form on $\mathcal{V}_\rho$ denoted by L_ρ such that

$$J_\rho(u_0 + h) - J_\rho(u_0) = \Re L_\rho(h) + o(\|h\|_{\mathcal{V}_\rho}). \tag{5}$$

The notation $\Re z$ stands for the real part of the complex number z. Furthermore, we assume that, for all $\rho \geq 0$, the adjoint problem

$$\begin{cases} v_\rho \in \mathcal{V}_\rho, \\ a_\rho(u, v_\rho) = -L_\rho(u) \qquad \forall u \in \mathcal{V}_\rho \end{cases} \tag{6}$$

admits an unique solution. We make the two following additional hypotheses.

Hypothesis 2 *There exist two complex numbers δ_a and δ_l and a function $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that, when ρ tends to zero,*

$$\begin{aligned} f(\rho) &\longrightarrow 0, \\ (a_\rho - \tilde{a}_\rho)(u_0, v_\rho) &= f(\rho)\delta_a + o(f(\rho)), \\ (l_\rho - \tilde{l}_\rho)(v_\rho) &= f(\rho)\delta_l + o(f(\rho)). \end{aligned}$$

Hypothesis 3 *There exists a real number δ_J such that*

$$J_\rho(u_\rho) - J_0(u_0) = \Re L_\rho(u_\rho - u_0) + f(\rho)\delta_J + o(f(\rho)).$$

Then, the asymptotic expansion of $j(\rho)$ is given by the theorem below.

Theorem 1 *If hypotheses 1, 2 and 3 hold, then*

$$j(\rho) - j(0) = f(\rho)\Re(\delta_a - \delta_l + \delta_J) + o(f(\rho)).$$

Proof. Using Equation (3) and Hypothesis 1, we obtain

$$\begin{aligned} j(\rho) - j(0) = {} & J_\rho(u_\rho) - J_0(u_0) + \Re(a_\rho - \tilde{a}_\rho)(u_0, v_\rho) + \Re a_\rho(u_\rho - u_0, v_\rho) \\ & -\Re(l_\rho - \tilde{l}_\rho)(v_\rho). \end{aligned}$$

Hypotheses 2 and 3 and Equation (6) yield

$$\begin{aligned} j(\rho) - j(0) = {} & \Re L_\rho(u_\rho - u_0) + f(\rho)\delta_J + f(\rho)\Re\delta_a - \Re L_\rho(u_\rho - u_0) \\ & -f(\rho)\Re\delta_l + o(f(\rho)), \end{aligned}$$

from which we deduce the announced result. □

3 The topological sensitivity problem

3.1 Problem formulation

Let Ω be an open, bounded and connected subset of $\mathbb{R}^N$, $N = 2$ or 3, with smooth boundary Γ. We assume for simplicity that Γ is piecewise of class $\mathcal{C}^\infty$, but this hypothesis could be considerably weakened. We consider a function $u_0 \in H^1(\Omega)$ satisfying the state equations

$$\begin{cases} \Delta u_0 + k^2 u_0 = 0 & \text{in } \Omega, \\ \partial_n u_0 = S u_0 + \sigma & \text{on } \Gamma. \end{cases} \tag{7}$$

Here, $\mathbf{n}$ denotes the outward unit normal of Γ, $k \in \mathbb{C}$, $S \in \mathcal{L}(H^{1/2}(\Gamma), H^{-1/2}(\Gamma))$, namely the space of continuous linear maps from $H^{1/2}(\Gamma)$ to $H^{-1/2}(\Gamma)$, and $\sigma \in H^{-1/2}(\Gamma)$. For a given $x_0 \in \Omega$ and a small parameter $\rho > 0$, we denote by Ω_ρ the perturbed domain. We shall address two situations.

- In the case of a perforation, we consider a fixed open and bounded subset ω of $\mathbb{R}^N$ containing the origin and whose boundary Σ is the union of two graphs of functions of class $\mathcal{C}^1$ from $\mathbb{R}^{N-1}$ into $\mathbb{R}$ (this technical hypothesis could also be weakened). We define $\omega_\rho = x_0 + \rho\omega$, $\Sigma_\rho = \partial\omega_\rho$ and $\Omega_\rho = \Omega \setminus \overline{\omega_\rho}$ (see Figure 1 (a)).
- In the case of the creation of a crack, we consider a bounded manifold Σ of dimension $N - 1$ which can be represented as the graph of a function of class $\mathcal{C}^1$ from $\mathbb{R}^{N-1}$ into $\mathbb{R}$. We define $\Sigma_\rho = x_0 + \rho\Sigma$ and $\Omega_\rho = \Omega \setminus \overline{\Sigma_\rho}$ (see Figure 1 (b)).

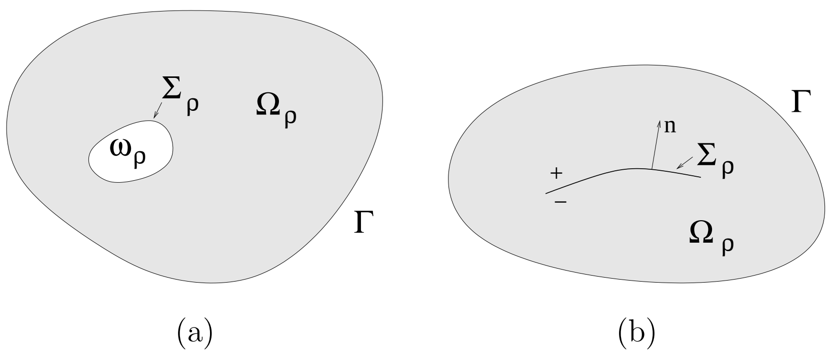


Figure 1. The perturbed domain: (a) perforated domain, (b) cracked domain.

Possibly changing the coordinate system, we suppose henceforth that $x_0 = 0$. In both cases, the new function $u_\rho \in H^1(\Omega_\rho)$ is assumed to solve the system

$$\begin{cases} \Delta u_\rho + k^2 u_\rho = 0 & \text{in } \Omega_\rho, \\ \partial_n u_\rho = S u_\rho + \sigma & \text{on } \Gamma, \\ \partial_n u_\rho = 0 & \text{on } \Sigma_\rho. \end{cases} \tag{8}$$

3.2 Well-posedness

The variational formulation of System (8) writes in the standard form (3) with

$$\mathcal{V}_\rho = H^1(\Omega_\rho),$$

$$\begin{cases} a_\rho(u,v) = \int_{\Omega_\rho} (\nabla u.\overline{\nabla v} - k^2 u\bar{v})dx - \int_\Gamma S u\bar{v}ds & \forall u, v \in \mathcal{V}_\rho, \\ l_\rho(v) = \int_\Gamma \sigma\bar{v}ds & \forall v \in \mathcal{V}_\rho. \end{cases} \tag{9}$$

For the sake of readability, the duality pairing between $H^{-\frac{1}{2}}(\Gamma)$ and $H^{\frac{1}{2}}(\Gamma)$ is denoted by an integral. The bar stands for the complex conjugate. This formulation applies also to Problem (7) when $\rho = 0$, with $\Omega_0 = \Omega$.

To insure well-posedness, we suppose that S verifies the following hypothesis.

Hypothesis 4 *The operator S is split into $S = S_0 + S_1$ where*

- $S_0 \in \mathcal{L}(H^{\frac{1}{2}}(\Gamma), H^{-\frac{1}{2}}(\Gamma))$ *and satisfies*

(1) $\int_\Gamma S_0\varphi\bar{\psi}ds = \overline{\int_\Gamma S_0\psi\bar{\varphi}ds} \qquad \forall\varphi,\psi \in H^{\frac{1}{2}}(\Gamma)$,

(2) $\int_\Gamma S_0\varphi\bar{\varphi}ds \leq 0 \qquad \forall\varphi \in H^{\frac{1}{2}}(\Gamma)$,

(3) $\left\{\int_\Gamma S_0\varphi\bar{\varphi}ds = 0\right\} \Rightarrow \{\varphi = 0$ *on a piece of nonzero measure of* $\Gamma\}$,

- $S_1 \in \mathcal{L}(H^{\frac{1}{2}}(\Gamma), H^{-\frac{1}{2}}(\Gamma))$ *and is a compact operator.*

Here are two examples of such an operator.

- When Ω is a disc (2D case), $k \in \{k \in \mathbb{C}, \Im k < 0\} \cup \mathbb{R}_+^*$ and S is the Dirichlet-to-Neumann operator corresponding to the Helmholtz equation on $\mathbb{R}^2 \setminus \overline{\Omega}$ with Sommerfeld condition at infinity, it is proved in [17] that Hypothesis 4 holds.
- If the boundary condition on Γ is of the form $\partial_n u - ik'u = \Phi$, where $k' \in \mathbb{C}$ (transmission condition), then $Su = ik'u$ and Hypothesis 4 is automatically checked by setting $S_0 = 0$.

We assume moreover the following uniqueness property, which is satisfied in the above cases (see *e.g.* [17] and [23]).

Hypothesis 5 *There exists $\rho_0 > 0$ such that for all $\rho \le \rho_0$,*

$$\{a_\rho(u,v) = 0 \ \forall v \in \mathcal{V}_\rho\} \Rightarrow \{u = 0\}\,,$$

$$\{a_\rho(u,v) = 0 \ \forall u \in \mathcal{V}_\rho\} \Rightarrow \{v = 0\}\,.$$

We consider a cost function J_ρ differentiable in the sense of Equation (5). The proof of the following proposition uses standard arguments (see *e.g.* [23]).

Proposition 1 *If Hypotheses 4 and 5 are satisfied, then for all $\rho \le \rho_0$*

- *the sesquilinear form a_ρ satisfies the inf-sup conditions*

$$\inf_{u\neq 0} \sup_{v\neq 0} \frac{|a_\rho(u,v)|}{\|u\|_{\mathcal{V}_\rho}\|v\|_{\mathcal{V}_\rho}} > 0, \qquad \inf_{v\neq 0} \sup_{u\neq 0} \frac{|a_\rho(u,v)|}{\|u\|_{\mathcal{V}_\rho}\|v\|_{\mathcal{V}_\rho}} > 0,$$

- *Problem (3) and Problem (6) are uniquely solvable.*

Remark 2 *The boundary condition on Γ adopted here has been chosen merely as an example. It could be replaced without any influence on the topological asymptotic analysis by any condition insuring that Problems (3) and (6) are well-posed.*

We wish now to apply Theorem 1 in this context. The imbedding $\mathcal{V}_0 \subset \mathcal{V}_\rho$ is defined by the restriction $u \in \mathcal{V}_0 \mapsto u_{|\Omega_\rho} \in \mathcal{V}_\rho$. To simplify the writing, the function $u_{|\Omega_\rho}$ will be still denoted by u. The analysis will be carried out in three steps:

(1) define $\tilde{a}_\rho$ and $\tilde{l}_\rho$ such that Hypothesis 1 holds,
(2) determine the function $f(\rho)$ and the complex numbers δ_a and δ_l such that Hypothesis 2 holds,
(3) for some examples of cost function, determine δ_J such that Hypothesis 3 holds.

For the first two points, the cases of a perforation and of a crack will be studied separately (Sections 4 and 5). Then, according to Theorem 1, the topological gradient at the origin will be

$$g(0) = \Re(\delta_a - \delta_l + \delta_J).$$

A shift of the coordinate system will provide immediately $g(x)$ for any $x \in \Omega$.

4 Creation of a hole

4.1 Formulation of the initial problem in the perforated domain

We focus on the case of a perforation, *i.e.*, $\Omega_\rho = \Omega \setminus \overline{\omega_\rho}$, $\Sigma_\rho = \partial\omega_\rho$. For all $\rho \geq 0$, we define the sesquilinear form

$$b_\rho(u,v) = \int_{\omega_\rho} (\nabla u.\overline{\nabla v} - k^2 u\bar{v})dx \qquad \forall u,v \in H^1(\omega_\rho).$$

Using the Poincaré inequality, it is easy to check that, when ρ is sufficiently small, namely k diam $(\omega_\rho) < 1$, b_ρ is coercive on $H^1_0(\omega_\rho)$. For such a ρ and $\varphi \in H^{1/2}(\Sigma_\rho)$, let $h^\varphi_\rho \in H^1(\omega_\rho)$ be the solution of

$$\begin{cases} \Delta h^\varphi_\rho + k^2 h^\varphi_\rho = 0 & \text{in } \omega_\rho, \\ h^\varphi_\rho = \varphi & \text{on } \Sigma_\rho. \end{cases} \tag{10}$$

We set for all $u, v \in H^1(\Omega_\rho)$

$$\begin{cases} \tilde{a}_\rho(u,v) = a_\rho(u,v) + b_\rho(h^u_\rho, h^v_\rho), \\ \tilde{l}_\rho(v) = l_\rho(v). \end{cases}$$

In the notation h^u_ρ, the letter u has been kept for simplicity to denote the trace of u on Σ_ρ. It is easy to check that Equation (4) holds.

Since $\tilde{l}_\rho = l_\rho$, we have by construction $\delta_l = 0$. For obtaining the general expression of the topological asymptotic, it remains to determine the first order expansion with respect to ρ of the quantity $(a_\rho - \tilde{a}_\rho)(u_0, v_\rho)$.

4.2 Preliminary calculus

We set

$$w_\rho = v_\rho - v_0.$$

In order to estimate w_ρ, we need the following assumption on the right hand side of the adjoint equation.

Hypothesis 6 *There exists a function L of regularity $\mathcal{C}^0 \cap H^1$ in the vicinity of the origin such that for all $u \in H^1(\Omega)$ and for all ρ small enough,*

$$L_0(u) = L_\rho(u_{|\Omega_\rho}) + \int_{\omega_\rho} L u dx. \tag{11}$$

Let $S^* \in \mathcal{L}(H^{1/2}(\Gamma), H^{-1/2}(\Gamma))$ be the adjoint operator of S, defined by

$$\int_\Gamma S\varphi\bar{\psi}ds = \overline{\int_\Gamma S^*\psi\bar{\varphi}ds} \qquad \forall \varphi, \psi \in H^{\frac{1}{2}}(\Gamma). \tag{12}$$

The function w_ρ solves:

$$\begin{cases} \Delta w_\rho + \bar{k}^2 w_\rho = 0 & \text{in } \Omega_\rho, \\ \partial_n w_\rho = -\partial_n v_0 & \text{on } \Sigma_\rho, \\ \partial_n w_\rho = S^* w_\rho & \text{on } \Gamma. \end{cases} \tag{13}$$

For all $\rho \geq 0$ and $\varphi \in H^{1/2}(\Sigma_\rho)$, we define the function $l_\rho^\varphi \in H^1(\omega_\rho)$ as the solution to

$$\begin{cases} \Delta l_\rho^\varphi = 0 & \text{in } \omega_\rho, \\ l_\rho^\varphi = \varphi & \text{on } \Sigma_\rho. \end{cases}$$

Some manipulations based on the Green formula and the equality $h_\rho^{u_0} = u_0$ lead to:

$$\begin{aligned} (a_\rho - \tilde{a}_\rho)(u_0, v_\rho) = &- \int_{\Sigma_\rho} \overline{\partial_n v_0}(u_0 - u_0(0))ds + \int_{\omega_\rho} L(u_0 - u_0(0))dx \\ &+ k^2 u_0(0) \int_{\omega_\rho} \overline{v_0} dx - \int_{\Sigma_\rho} \overline{\partial_n l_\rho^{w_\rho}}(u_0 - u_0(0))ds + k^2 \int_{\omega_\rho} u_0 \overline{l_\rho^{w_\rho}} dx. \end{aligned} \tag{14}$$

4.3 Asymptotic analysis

4.3.1 Approximation of w_ρ

A suitable approximation of w_ρ is expected to be provided by the function

$$p_\rho(x) = \rho P\left(\frac{x}{\rho}\right),$$

where the function $P \in W^1(\mathbb{R}^N \setminus \bar{\omega})$, independent of ρ, is the solution of

$$\begin{cases} \Delta P = 0 & \text{in } \mathbb{R}^N \setminus \bar{\omega}, \\ P = O(1/r^{N-1}) & \text{at } \infty, \\ \partial_n P(x) = -\nabla v_0(0).\mathbf{n} & \text{on } \Sigma. \end{cases} \tag{15}$$

We recall that, for an exterior domain $\omega' = \mathbb{R}^N \setminus \bar{\omega}$, the Sobolev space $W^1(\omega')$ is defined by (see *e.g.* [16,13,9]):

$$W^1(\omega') = \left\{ u \in \mathcal{D}'(\omega'), \frac{u}{(1+r)\ln r} \in L^2(\omega') \text{ and } \nabla u \in L^2(\omega') \right\} \qquad \text{in 2D},$$

$$W^1(\omega') = \left\{ u \in \mathcal{D}'(\omega'), \frac{u}{1+r} \in L^2(\omega') \text{ and } \nabla u \in L^2(\omega') \right\} \qquad \text{in 3D.}$$

The decay $P = O(1/r^{N-1})$ stems from the fact that $\nabla v_0(0).\mathbf{n}$ has zero mean. The function P can be written as the single layer potential:

$$P(x) = \int_\Sigma \lambda(y) E(x-y) ds(y) \qquad \forall x \in \mathbb{R}^N \setminus \bar{\omega}, \tag{16}$$

where the density $\lambda \in H_0^{-1/2}(\Sigma)$ is the unique solution of the boundary integral equation

$$\frac{\lambda(x)}{2} + \int_\Sigma \lambda(y) \partial_{n_x} E(x-y) ds(y) = -\nabla v_0(0).\mathbf{n} \qquad \forall x \in \Sigma. \tag{17}$$

We recall the expression of the fundamental solution of the Laplace operator

$$E(x) = \begin{cases} \dfrac{1}{2\pi} \ln |x| \text{ in 2D}, \\ -\dfrac{1}{4\pi |x|} \quad \text{in 3D}, \end{cases}$$

and the definition of the space

$$H_0^{-\frac{1}{2}}(\partial\omega) = \left\{ u \in H^{-\frac{1}{2}}(\partial\omega), \int_{\partial\omega} u ds = 0 \right\}.$$

4.3.2 Topological asymptotic expansion

First, we write Equation (14) in the form

$$\begin{aligned}(a_\rho - \tilde{a}_\rho)(u_0, v_\rho) = &- \int_{\Sigma_\rho} \overline{\partial_n v_0} (u_0 - u_0(0)) ds + k^2 \rho^N |\omega| u_0(0) \overline{v_0(0)} \\ &- \int_{\Sigma_\rho} \overline{\partial_n l_\rho^{p_\rho}} (u_0 - u_0(0)) ds + \sum_{i=1}^4 \mathcal{E}_i(\rho),\end{aligned}$$

with

$$\mathcal{E}_1(\rho) = -\int_{\Sigma_\rho} (\overline{\partial_n l_\rho^{w_\rho}} - \overline{\partial_n l_\rho^{p_\rho}})(u_0 - u_0(0)) ds, \quad \mathcal{E}_2(\rho) = \int_{\omega_\rho} L(u_0 - u_0(0)) dx,$$

$$\mathcal{E}_3(\rho) = k^2 u_0(0) \left[\int_{\omega_\rho} \overline{v_0} dx - \rho^N |\omega| \overline{v_0(0)} \right], \quad \mathcal{E}_4(\rho) = k^2 \int_{\omega_\rho} u_0 \overline{l_\rho^{w_\rho}} dx.$$

Above, we have denoted by $|\omega|$ the Lebesgue measure of ω, *i.e.*, $|\omega| = 4\pi/3$ in 3D, $|\omega| = \pi$ in 2D. To improve the readability, all error estimates are reported in Section 8. For all $\varphi \in H^{1/2}(\Sigma)$, let l^φ denote the solution of

$$\begin{cases} \Delta l^\varphi = 0 \quad \text{in} \quad \omega, \\ \quad l^\varphi = \varphi \quad \text{on} \quad \Sigma. \end{cases}$$

For all $x \in \Sigma_\rho$, we have

$$l_\rho^{p_\rho}(x) = \rho l^P\left(\frac{x}{\rho}\right) \qquad \text{and} \qquad \partial_n l_\rho^{p_\rho}(x) = \partial_n l^P\left(\frac{x}{\rho}\right).$$

The jump relation of the single layer potential yields

$$\lambda(y) = -\nabla v_0(0).\mathbf{n} - \partial_n l^P(y) \qquad \forall y \in \Sigma.$$

Thus, we can write

$$\begin{aligned}
(a_\rho - \tilde{a}_\rho)(u_0, v_\rho) &= \int_{\Sigma_\rho} \overline{\lambda\left(\frac{x}{\rho}\right)}(u_0 - u_0(0))ds + k^2\rho^N|\omega|u_0(0)\overline{v_0(0)} \\
&\quad - \int_{\Sigma_\rho}(\overline{\partial_n v_0} - \overline{\nabla v_0(0).\mathbf{n}})(u_0 - u_0(0))ds + \sum_{i=1}^4 \mathcal{E}_i(\rho) \\
&= \rho^{N-1}\int_\Sigma \overline{\lambda(x)}(u_0(\rho x) - u_0(0))ds + k^2\rho^N|\omega|u_0(0)\overline{v_0(0)} \\
&\quad -\rho^{N-1}\int_\Sigma(\overline{\partial_n v_0(\rho x)} - \overline{\nabla v_0(0).\mathbf{n}})(u_0(\rho x) - u_0(0))ds + \sum_{i=1}^4 \mathcal{E}_i(\rho).
\end{aligned}$$

Finally, denoting

$$\begin{aligned}
\mathcal{E}_5(\rho) &= -\rho^{N-1}\int_\Sigma(\overline{\partial_n v_0(\rho x)} - \overline{\nabla v_0(0).\mathbf{n}})(u_0(\rho x) - u_0(0))ds, \\
\mathcal{E}_6(\rho) &= \rho^{N-1}\int_\Sigma \overline{\lambda(y)}(u_0(\rho y) - u_0(0) - \nabla u_0(0).\rho y)ds(y),
\end{aligned}$$

we obtain

$$(a_\rho - \tilde{a}_\rho)(u_0, v_\rho) = \rho^N \nabla u_0(0).\int_\Sigma \overline{\lambda(x)}x ds(x) + k^2\rho^N|\omega|u_0(0)\overline{v_0(0)} + \sum_{i=1}^6 \mathcal{E}_i(\rho).$$

In Subsection 8.2, we prove that $|\mathcal{E}_i(\rho)| = o(\rho^N)$ for all $i = 1, ..., 6$. Therefore, we have

$$f(\rho) = \rho^N,$$

$$\delta_a = \nabla u_0(0).\int_\Sigma \overline{\lambda(x)}x ds(x) + k^2|\omega|u_0(0)\overline{v_0(0)}.$$

We are going to rewrite this expression in order to show the dependence of δ_a with respect to the adjoint state. Thanks to the linearity of Equation (17), it comes

$$\int_\Sigma \lambda(x)x ds(x) = -\mathcal{A}\nabla v_0(0)$$

where the matrix $\mathcal{A}$ is defined by

$$\mathcal{A}V = \int_\Sigma \eta(x)x ds(x) \qquad \forall V \in \mathbb{C}^N, \tag{18}$$

the density $\eta \in H_0^{-1/2}(\Sigma)$ being the unique solution of

$$\frac{\eta(x)}{2} + \int_\Sigma \eta(y)\partial_{n_x} E(x-y)ds(y) = V.\mathbf{n} \qquad \forall x \in \Sigma. \tag{19}$$

Since the matrix $\mathcal{A}$ maps a vector of $\mathbb{R}^N$ to a vector of $\mathbb{R}^N$, its coefficients are real numbers. It corresponds to the well-known notion of polarization tensor [22]. It is proved *e.g.* in [11] that $\mathcal{A}$ is symmetric positive definite in our context. We refer to [2,4] for further properties of polarization tensors. We derive the following result as a consequence of Theorem 1.

Theorem 2 *We assume that*

- *the cost function satisfies Hypothesis 3 with $f(\rho) = \rho^N$,*
- *Hypotheses 4, 5 and 6 are satisfied,*
- *the adjoint state v_0 solves*

$$\begin{cases} v_0 \in H^1(\Omega), \\ a_0(u, v_0) = -L_0(u) \qquad \forall u \in H^1(\Omega), \end{cases} \tag{20}$$

- *the polarization tensor $\mathcal{A}$ is defined by (18).*

Then the cost function admits the asymptotic expansion:

$$j(\rho) - j(0) = \rho^N \Re\left(-\nabla u_0(0).\mathcal{A}\overline{\nabla v_0(0)} + k^2|\omega|u_0(0)\overline{v_0(0)} + \delta_J\right) + o(\rho^2). \tag{21}$$

4.4 Spherical hole

The case where $\omega = B(0,1)$, the unit ball of $\mathbb{R}^N$, is of particular interest for the applications. By using spherical (polar in 2D) coordinates, or by solving explicitly the associated exterior and interior problems and by calculating the density as the jump between the normal derivatives, one can check that the solution of Equation (19) is

$$\eta(x) = \frac{N}{N-1}V.x \qquad \forall x \in \Sigma$$

and consequently that

$$\mathcal{A} = \frac{N}{N-1}|\omega|I.$$

The topological asymptotic reads

$$j(\rho)-j(0)=|\omega|\rho^N\Re\left(-\frac{N}{N-1}\nabla u_0(0).\overline{\nabla v_0(0)}+k^2u_0(0)\overline{v_0(0)}+\frac{\delta_J}{|\omega|}\right)+o(\rho^N). \tag{22}$$

5 Creation of a Crack

5.1 Formulation of the initial problem in the cracked domain

We have here $\Omega_\rho=\Omega\setminus\overline{\Sigma_\rho}$. We set for all $\rho\geq 0$ and all $u,v\in H^1(\Omega_\rho)$

$$\begin{cases}\tilde{a}_\rho(u,v)=a_\rho(u,v),\\ \tilde{l}_\rho(v)=a_\rho(u_0,v).\end{cases}$$

It is then obvious that Hypothesis 1 holds. We have in this case by construction $\delta_a=0$, and we shall determine $f(\rho)$ and δ_l.

5.2 Preliminary calculus

We obtain thanks to the Green formula

$$\begin{aligned}(l_\rho-\tilde{l}_\rho)(v_\rho)&=\int_{\Sigma_\rho}\partial_n u_0[\overline{w_\rho}]ds\\ &=\rho^{N-1}\int_\Sigma \partial_n u_0(\rho x)[\overline{w_\rho(\rho x)}]ds,\end{aligned}$$

where $[v_\rho]=v^+_{\rho|\Sigma_\rho}-v^-_{\rho|\Sigma_\rho}\in H^{1/2}_{00}(\Sigma_\rho)$ (see Figure 1) and $w_\rho=v_\rho-v_0$. We make the following assumption on the cost function.

Hypothesis 7 *For all ρ sufficiently small and all $u\in H^1(\Omega)$,*

$$L_0(u)=L_\rho(u_{|\Omega_\rho}). \tag{23}$$

Moreover, L_0, as a distribution, is of regularity H^1 in a neighborhood of the origin.

Thus, the function w_ρ satisfies:

$$\begin{cases} \Delta w_\rho + \bar{k}^2 w_\rho = 0 & \text{in } \Omega_\rho, \\ \partial_n w_\rho = -\partial_n v_0 & \text{on } \Sigma_\rho, \\ \partial_n w_\rho = S^* w_\rho & \text{on } \Gamma. \end{cases} \tag{24}$$

5.3 Asymptotic analysis

5.3.1 Approximation of w_ρ

We will show that a suitable approximation of w_ρ is provided by the function

$$p_\rho(x) = \rho P_\rho\left(\frac{x}{\rho}\right),$$

where $P_\rho \in W^1(\mathbb{R}^N \setminus \Sigma)$ is the solution of

$$\begin{cases} \Delta P_\rho = 0 & \text{in } \mathbb{R}^2 \setminus \overline{\Sigma}, \\ P_\rho = O(1/r^{N-1}) & \text{at } \infty, \\ \partial_n P_\rho(x) = -\partial_n v_0(\rho x) & \text{on } \Sigma. \end{cases}$$

This function P_ρ can be written with the help of the double layer potential:

$$P_\rho(x) = \int_\Sigma \mu_\rho(y) \partial_{n_y} E(x-y) ds(y) \qquad \forall x \in \mathbb{R}^N \setminus \Sigma. \tag{25}$$

The density $\mu_\rho \in H_{00}^{1/2}(\Sigma)$ is defined by

$$\mu_\rho = T(-\partial_n v_0(\rho x)), \tag{26}$$

where T is a continuous isomorphism from $H_{00}^{1/2}(\Sigma)'$ into $H_{00}^{1/2}(\Sigma)$. We recall the expression of the map T^{-1} for a linear (planar in 3D) crack (see [20,9]): for all $\mu \in (H_{00}^{1/2} \cap \mathcal{C}^1)(\Sigma)$ and $\varphi \in \mathcal{D}(\Sigma)$,

$$< T^{-1}\mu, \varphi >= -\int_\Sigma \int_\Sigma \frac{d\mu}{ds}(x) \frac{d\varphi}{ds}(y) E(x-y) ds(x) ds(y) \qquad \text{in 2D},$$

$$< T^{-1}\mu, \varphi >= -\int_\Sigma \int_\Sigma \mathrm{curl}_\Sigma\, \mu(x).\mathrm{curl}_\Sigma\, \varphi(y) E(x-y) ds(x) ds(y) \qquad \text{in 3D}.$$

In this latter expression, we use the notation

$$\mathrm{curl}_\Sigma\, u = \mathbf{n} \times \nabla \tilde{u},$$

where $\tilde{u}$ is an arbitrary lifting of u in $\mathbb{R}^N \setminus \overline{\Sigma}$.

Then, we approximate μ_ρ by

$$\mu = T(-\nabla v_0(0).\mathbf{n}). \tag{27}$$

5.3.2 Topological asymptotic expansion

Denoting by

$$\mathcal{E}_1(\rho) = \rho^{N-1} \int_\Sigma \partial_n u_0(\rho x)[\overline{(w_\rho - p_\rho)(\rho x)}]ds,$$

we have

$$(l_\rho - \tilde{l}_\rho)(v_\rho) = \rho^N \int_\Sigma \partial_n u_0(\rho x)[\overline{P_\rho}]ds + \mathcal{E}_1(\rho).$$

By virtue of the jump relation of the double layer potential, $[P_\rho] = -\mu_\rho$. Hence,

$$\begin{aligned}(l_\rho - \tilde{l}_\rho)(v_\rho) &= -\rho^N \int_\Sigma \partial_n u_0(\rho x)\overline{\mu_\rho}ds + \mathcal{E}_1(\rho) \\ &= -\rho^N \int_\Sigma \partial_n u_0(\rho x)\overline{\mu}ds + \mathcal{E}_1(\rho) + \mathcal{E}_2(\rho)\end{aligned}$$

with

$$\mathcal{E}_2(\rho) = -\rho^N \int_\Sigma \partial_n u_0(\rho x)(\overline{\mu_\rho - \mu})ds.$$

Finally, setting

$$\mathcal{E}_3(\rho) = -\rho^N \int_\Sigma (\partial_n u_0(\rho x) - \nabla u_0(0).\mathbf{n})\overline{\mu}ds,$$

we get

$$(l_\rho - \tilde{l}_\rho)(v_\rho) = -\rho^N \int_\Sigma \nabla u_0(0).\mathbf{n}\overline{\mu}ds + \sum_{i=1}^{3} \mathcal{E}_i(\rho).$$

In Subsection 8.3, we prove that $|\mathcal{E}_i(\rho)| = o(\rho^N)$ for all $i = 1, ..., 3$. Therefore, we have

$$f(\rho) = \rho^N,$$

$$\delta_l = \int_\Sigma \nabla u_0(0).\mathbf{n}\overline{\mu}ds.$$

Again, it is convenient to introduce the polarization matrix $\mathcal{B}$ defined by

$$\mathcal{B}V = \int_\Sigma \eta \mathbf{n} ds \qquad \forall V \in \mathbb{C}^N, \tag{28}$$

where

$$\eta = T(V.\mathbf{n}). \tag{29}$$

We obtain the following result as a consequence of Theorem 1.

Theorem 3 *We assume that*

- *the cost function satisfies Hypothesis 3 with* $f(\rho) = \rho^N$,
- *Hypotheses 4, 5 and 7 are satisfied,*
- *the adjoint state* v_0 *is solution of (20),*
- *the polarization tensor* $\mathcal{B}$ *is defined by (28).*

Then the cost function admits the asymptotic expansion:

$$j(\rho) - j(0) = \rho^N \Re\left(-\nabla u_0(0).\mathcal{B}\overline{\nabla v_0(0)} + \delta_J\right) + o(\rho^N). \tag{30}$$

5.4 Linear and planar cracks

- *Linear crack (2D).* Let Σ be the line segment $\{(s,0), -1 < s < 1\}$. The solution of Equation (29) is

$$\eta(x) = 2\sqrt{1-s^2}(V.\mathbf{n}) \quad \forall x = (s,0) \in \Sigma.$$

- *Planar circular crack (3D).* Consider the planar unit disc $\Sigma = \{(r\cos\theta, r\sin\theta, 0), 0 \le r < 1, 0 \le \theta < 2\pi\}$. One can check (by a very technical calculus) that the corresponding density is

$$\eta(x) = \frac{4}{\pi}\sqrt{1-r^2}(V.\mathbf{n}) \quad \forall x \in \Sigma,\ |x| = r.$$

The integration over Σ of the above densities leads to the polarization matrix

$$\mathcal{B} = \alpha \mathbf{n} \otimes \mathbf{n},$$

where $\mathbf{n} \otimes \mathbf{n} := \mathbf{n}\mathbf{n}^T$ and

$$\alpha = \begin{cases} \pi & \text{in 2D,} \\ \frac{8}{3} & \text{in 3D.} \end{cases}$$

The topological asymptotic expansion reads

$$j(\rho) - j(0) = \rho^N \Re\left(-\alpha(\nabla u_0(0).\mathbf{n})(\overline{\nabla v_0(0).\mathbf{n}}) + \delta_J\right) + o(\rho^N). \tag{31}$$

5.5 General comments

For an arbitrarily shaped hole or crack, the topological gradient expresses by means of a polarization tensor that can be computed numerically. When the principal part of the differential operator is different from the Laplacian, the adequate fundamental solution must be used for solving the integral equations (19) and (29). The term δ_J, that depends on the chosen criterion, is explicited for some particular choices in the following section.

6 Particular cost functions

The following theorem is proved in Subsection 8.4.

Theorem 4 *For the following cost functions, Hypotheses 3, 6 and 7 hold for the values of δ_J indicated below.*

(1) First example. The easiest case consists in a cost function of the form

$$J_\rho(u_\rho) = J(u_{\rho|D_R})$$

where $D_R = \Omega \backslash \overline{B(0,R)}$, R being a fixed radius such that $\overline{B(0,R)} \subset \Omega$. We assume that there exists $L_0 \in (H^1(D_R))'$ such that, when $h \in H^1(D_R)$,

$$J(u_{0|D_R} + h) - J(u_{0|D_R}) = \Re L_0(h) + O(\|h\|^2_{1,D_R}).$$

For such a criterion, we have $\delta_J = 0$.

(2) Second example. It consists in the least-square cost function

$$J_\rho(u) = \int_{\Omega_\rho} |u - u_d|^2 dx,$$

where u_d belongs to $H^1(\Omega)$ and is continuous in the vicinity of the origin. In this case,

$$\delta_J = \begin{cases} -|\omega||u_0(0) - u_d(0)|^2 & \textit{for a hole,} \\ 0 & \textit{for a crack.} \end{cases}$$

7 Numerical experiments

7.1 Description of the problem and of the recovery method

It is of particular interest to apply the topological asymptotic approach to the equations of elastodynamics. Indeed many target detection methods involved in fields such as non destructive testing, submarine detection or medical imaging use the so-called pulse-echo method with acoustic or elastic waves at ultrasonic frequencies. The basic principle is the one of echography. A short pulse source is sent through the medium with an emitter/receiver apparatus and the variation of elastic properties of the medium (characterizing the kind of target) generates scattered waves that are recorded by the receiving apparatus. In the case of air bubbles, cracks or delaminations in solids, a Neumann boundary condition is involved at the edge of the defect. The major issue is to be able to read the results so as to detect, localize and characterize the target(s). The topological gradient is a great prospect for the automatic interpretation of these kind of results. It is clear that the pulse-echo method is intrinsically a transient phenomenon, then in order to mimic it we need to derive asymptotic formulas for the elastodynamics equations in the time domain.

To do so we extend the formulas obtained in the time-harmonic case to the dynamic problem by using the duality of the frequency and time domains through the Fourier transform. The time domain problem associated to the linear elasticity problem reads

$$\rho_d \frac{\partial^2 u}{\partial t^2} - \text{div } \sigma(u) = 0. \tag{32}$$

The corresponding time-harmonic problem is

$$-\rho_d \, \nu^2 \hat{u} - \text{div } \sigma(\hat{u}) = 0,$$

where $\hat{u}(x,\nu) = \int_{\mathbb{R}} u(t,x) e^{-i\nu t} dt$ is the Fourier transform of the displacement field $u(x,t)$. The notations ρ_d and ν standing for the material density and the pulsation, respectively, are adopted to avoid any confusion with the previously introduced notations ρ (the radius of the infinitesimal perforation) and ω (the hole of unitary size). We recall that in the context of linear elasticity, which is adequate for our applications, the stress tensor $\sigma(u)$ is a linear function of the first spatial derivatives of u, characterized by Hooke's tensor which is well-known to be symmetric positive definite. Hence we are in the scope of the analysis developed beforehand.

Starting with the cost function of the time domain problem and using successively Fubbini's theorem and Parseval's equality, it comes

$$J(u) = \frac{1}{2} \int_{\mathbb{R}} \Big(\int_{\Gamma_m} |u - u_m|^2 dx \Big) \, dt = \int_{\mathbb{R}} \Big(\frac{1}{2} \int_{\Gamma_m} |\hat{u} - \hat{u}_m|^2 dx \Big) \, d\nu = \int_{\mathbb{R}} J^\nu(\hat{u}(.,\nu)) \, d\nu. \tag{33}$$

Here, Γ_m denotes the sensors locations, namely a part of the border of Ω where the measurements are performed and u_m is the measured displacement field. At a given frequency, the topological asymptotic expansion for $J^\nu(\hat{u}(\cdot,\nu))$ is known. Starting from

$$J(u_\rho) - J(u_0) = \int_{\mathbb{R}} (J^\nu(\hat{u}_\rho(.,\nu)) - J^\nu(\hat{u}_0(.,\nu))) \, d\nu, \tag{34}$$

then using (33) and Parseval's equality, and assuming that $\int_{\mathbb{R}} o(\rho^2) \; d\nu \sim o(\rho^2)$, one obtains the expressions for the time domain problem. Denoting $\hat{u}_0 = \hat{u}_0(x_0,\nu)$ to simplify the writing, one has for instance for a circular hole created around the point x_0 (see Theorem 2 with the polarization tensor replaced by the one obtained by Garreau et al [12]):

$$
\begin{aligned}
& J(u_\rho) - J(u_0) \\
&= \pi\rho^2 \int_{\mathbb{R}} \Big(\beta\sigma(\hat{u}_0) : \varepsilon(\hat{v}_0) + \gamma \mathrm{tr}\sigma(\hat{u}_0)\mathrm{tr}\varepsilon(\hat{v}_0) + \rho_d\, \nu^2\, \hat{u}_0.\hat{v}_0\Big)\ d\nu + o(\rho^2) \qquad (35) \\
&= \pi\rho^2 \int_{\mathbb{R}} \left(\beta\sigma(u_0) : \varepsilon(v_0) + \gamma \mathrm{tr}\sigma(u_0)\mathrm{tr}\varepsilon(v_0) - \rho_d\, \partial_t u_0.\, \partial_t v_0\right)\ dt + o(\rho^2).
\end{aligned}
$$

where ε is the strain tensor of the material, and β, γ are combinations of the Lamé coefficients λ and μ. In plane stress $\beta = -8\frac{\lambda+\mu}{3\lambda+2\mu}$, $\gamma = -2\frac{(\lambda+\mu)(\lambda-2\mu)}{(\lambda+2\mu)(3\lambda+2\mu)}$. The topological gradient at any point $x_0 \in \Omega$ is then

$$
g(x_0) = \int_{\mathbb{R}} \left(\beta\sigma(u_0) : \varepsilon(v_0) + \gamma \mathrm{tr}\sigma(u_0)\mathrm{tr}\varepsilon(v_0) - \rho_d\, \partial_t u_0.\, \partial_t v_0\right)\ dt,
$$

where all the quantities in the integrand are evaluated at the point x_0. Practically we will not have access to the solutions for $t \in \mathbb{R}$, but only over an interval $[0, T]$. Then T must be taken large enough so that the amplitude of the fields in the computation domain after the time T is weak enough to be neglected when computing the topological gradient.

7.2 The forward solver

It can be shown that the adjoint problem can be solved with the forward solver provided attention is paid to the fact that the adjoint problem solves backward in time, from $t = T$ to $t = 0$.

We use a finite difference C++ code following Virieux's numerical scheme [27] which is accurate at the order 2 in space and time and intrinsically centered. It allows one to take into account abrupt ruptures of elastic properties or density such as fluid/solid interfaces. This code is integrated to the software ACEL developed by M. Tanter [26] and which is dedicated to the simulation of acoustic and elastic wave propagation. The boundary conditions at the edges of the computation domain are either of the classical Dirichlet and Neumann type, or of absorbing type to simulate unbounded propagation. The implemented absorbing conditions are Perfectly Matched Layers following Collino and Tsogka [8].

7.3 Numerical results

In this section we present numerical results relative to non destructive testing. The measurement step is up to now replaced by a numerical solving of the forward problem in the presence of the obstacles. The presented results are 2D since the 3D code is still being developed.

7.3.0.1 ***Unique defect in an isotropic solid***

The considered medium is an isotropic aluminium slab of density $\rho_d = 2572\,kg.m^{-3}$, the compressional (index p) and shear (index s) speeds of propagation are $v_p = 6408\,m.s^{-1}$ and $v_s = 3228\,m.s^{-1}$. The ultrasonic linear array is placed at the bottom of the slab. We use a 55 sensors array, all of them being used in emission and receive. Absorbing conditions are positioned at the boundaries of the computation domain, except at the bottom where a Dirichlet condition models the presence of the sensors.
The emitted signal is a pulse of 1 μs at the central frequency of 2 MHz (fig. 2). The defect is as shown on figure 3(a), it corresponds to a cylindrical hole whose

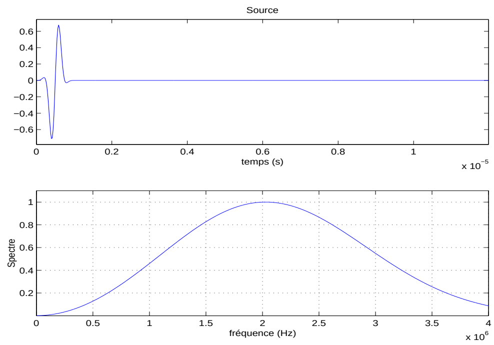
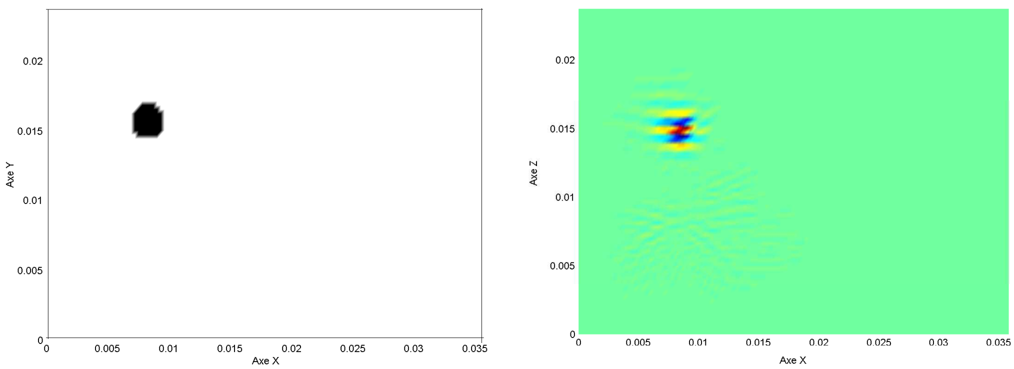


Figure 2. Source : temporal signal (top), frequency spectrum (bottom)

size is of the order of the compressional wavelength λ_p. Then the boundary condition at the edges of the defect is 2D Neumann.



(a) (b)

Figure 3. Detection of a unique defect. (a) Position of the defect, (b) Levels of the topological gradient

The position of the defect is clearly pointed out by the high level values of the topological gradient. The negative values (in red) indicate the bottom of the defect. Indeed, since we insonify from the bottom of the slab, it is clear that we have information about the shape of the bottom of the defect, and poor information in the acoustical shadow zones.

7.3.0.2 ***Multiple shaped defects*** Let us now test the ability of the method to detect multiple defects of different sizes and shapes. We put five defects of various shapes in the aluminium slab (fig. 4(a)). Their horizontal sizes vary from $\frac{\lambda_p}{5}$ to $\frac{3\lambda_p}{2}$. These defects are well resolved since they are separated from more than a wavelength. We use the same linear array and source as in the previous example. In order to draw nearer to experimental non destructive testing conditions, we have added white noise to the simulated measurements. Figures 4(b)(c)(d) show the levels of the topological gradient when the noise level is respectively of 0%, 5% and 10% of the maximum value of the emitted signal, corresponding respectively to signal to noise ratios of ∞, 5 and 2.5 on the signal scattered by the defects. In each presented result, the five defects are detected and localized. The approximate sizes and shapes of the obstacles are obtained, except in the shadow zones. It is very interesting to see that the method has a robust behavior upon addition of noise to the simulated measurements. It allows one to be optimistic as for the application of the method to experimental measurements that are intrinsically noisy.

8 Proofs

The aim of this section is to prove Theorems 2, 3 and 4. We recall that, for any fixed radius $R > 0$, $D_R = \Omega \setminus \overline{B(0,R)}$. The letter c denotes any positive constant that may change from place to place but that never depends on ρ.

8.1 Preliminary lemmas

The following lemmas are valid for both types of domain perturbation. We will use the notations:

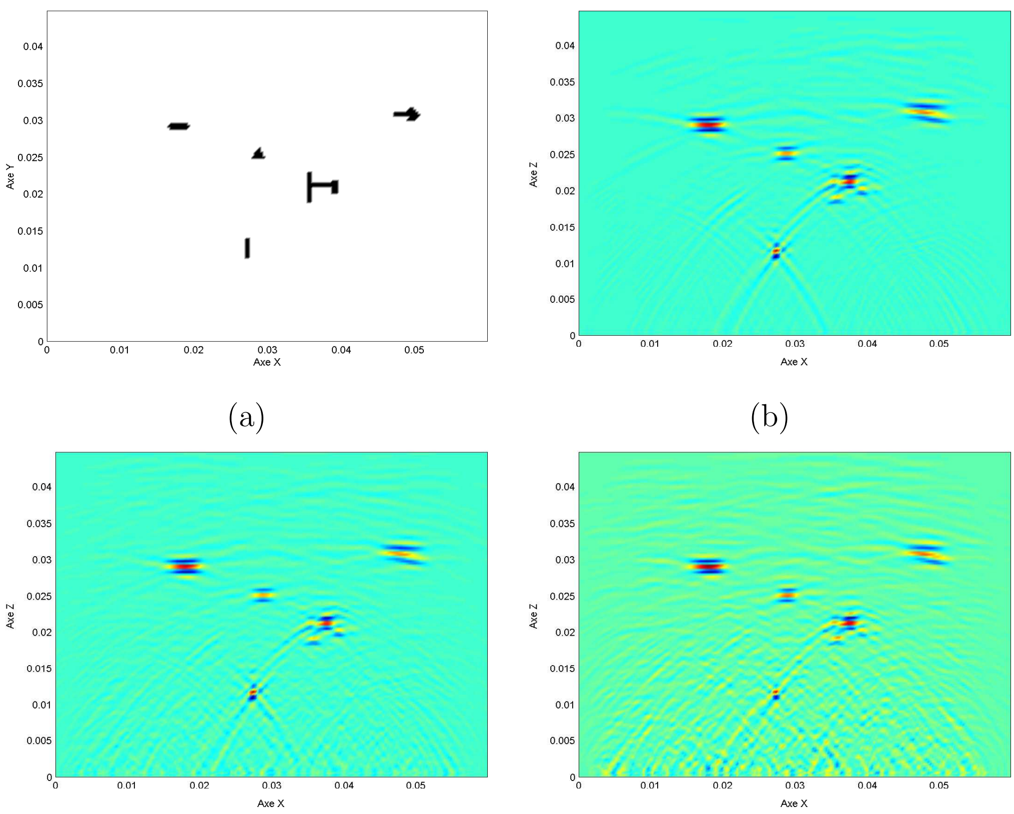
- for a perforation,

$$\mathcal{T}(\Sigma) = H_0^{-1/2}(\Sigma), \qquad \mathcal{T}(\Sigma_\rho) = H_0^{-1/2}(\Sigma_\rho),$$

$$\mathcal{U} = \mathbb{R}^N \setminus \overline{\omega}, \qquad \mathcal{U}_\rho = \mathbb{R}^N \setminus \overline{\omega_\rho},$$

- for a crack,

$$\mathcal{T}(\Sigma) = H_{00}^{1/2}(\Sigma)', \qquad \mathcal{T}(\Sigma_\rho) = H_{00}^{1/2}(\Sigma_\rho)',$$



(a) (b)

(c) (d)

Figure 4. Detection of multiple shaped defects. (a) Positions of the defects, (b)-(d) Levels of the topological gradient (b) with no added noise, (c) with 5% of noise, (d) with 10% of noise

$$\mathcal{U} = \mathbb{R}^N \setminus \overline{\Sigma}, \qquad \mathcal{U}_\rho = \mathbb{R}^N \setminus \overline{\Sigma_\rho}.$$

Lemma 3 *Consider $g \in \mathcal{T}(\Sigma)$ and let $z \in W^1(\mathcal{U})$ be the solution of the problem*

$$\begin{cases} \Delta z = 0 & \text{in } \mathcal{U}, \\ z = O(1/r^{N-1}) & \text{at } \infty, \\ \partial_n z = g & \text{on } \Sigma. \end{cases}$$

There exists $c > 0$ such that

$$|z|_{1,\frac{1}{\rho}D_R} \leq c\rho^{\frac{N}{2}} \|g\|_{\mathcal{T}(\Sigma)}.$$

Proof. Let us first consider the case of a hole. We have the single layer potential representation

$$z(x) = \int_\Sigma \lambda(y) E(x-y) ds(y), \qquad \forall x \in \mathbb{R}^N \setminus \bar{\omega},$$

where $\lambda \in H_0^{-1/2}(\Sigma)$ depends continuously on g. Using a Taylor expansion of E computed at the point x, we obtain that

$$|\nabla z(x)| \leq \frac{c}{|x|^N} \|g\|_{-1/2,\Sigma},$$

from which we deduce easily the desired estimate. For a crack, we use the double layer potential representation

$$z(x) = \int_{\Sigma} \mu(y) \partial_{n_y} E(x-y) ds(y), \qquad \forall x \in \mathbb{R}^N \setminus \overline{\Sigma},$$

with $\mu \in H_{00}^{1/2}(\Sigma)$. The reasoning is then similar to the previous case. □

Lemma 4 *For all ρ and all $g \in \mathcal{T}(\Sigma_\rho)$, the solution $z_\rho \in W^1(\mathcal{U}_\rho)$ to the problem*

$$\begin{cases} \Delta z_\rho = 0 & \text{in } \ \mathcal{U}_\rho, \\ z_\rho = O(1/r^{N-1}) & \text{at } \ \infty, \\ \partial_n z_\rho = g & \text{on } \ \Sigma_\rho \end{cases}$$

satisfies the estimates

$$\|z_\rho\|_{0,\Omega_\rho} \leq c\rho^{\frac{N}{2}+1} \|g(\rho x)\|_{\mathcal{T}(\Sigma)},$$

$$|z_\rho|_{1,\Omega_\rho} \leq c\rho^{\frac{N}{2}} \|g(\rho x)\|_{\mathcal{T}(\Sigma)},$$
$$|z_\rho|_{1,D_R} \leq c\rho^{N} \|g(\rho x)\|_{\mathcal{T}(\Sigma)}.$$

Proof. We set $Z_\rho(x) = z_\rho(\rho x)$. The function Z_ρ solves

$$\begin{cases} \Delta Z_\rho = 0 & \text{in } \ \mathcal{U}, \\ Z_\rho = O(1/r^{N-1}) & \text{at } \ \infty, \\ \partial_n Z_\rho = \rho g(\rho x) & \text{on } \ \Sigma. \end{cases}$$

By elliptic regularity, we have

$$\|Z_\rho\|_{W^1(\mathcal{U})} \leq c \|\rho g(\rho x)\|_{\mathcal{T}(\Sigma)}.$$

A change of variable yields

$$\|z_\rho\|_{0,\Omega_\rho} \leq c\rho^{\frac{N}{2}} \|\rho g(\rho x)\|_{\mathcal{T}(\Sigma)},$$

$$|z_\rho|_{1,\Omega_\rho} \leq c\rho^{\frac{N}{2}-1} \|\rho g(\rho x)\|_{\mathcal{T}(\Sigma)}.$$

The last inequality to be proved stems from Lemma 3 and a change of variable. □

A proof of the following lemma can be found in [5].

Lemma 5 *Consider $\sigma \in H^{-1/2}(\Gamma)$, $\rho \geq 0$, $f \in L^2(\Omega)$ and let $z_\rho \in H^1(\Omega_\rho)$ be the solution of the problem*

$$\begin{cases} \Delta z_\rho + k^2 z_\rho = f & in \ \ \Omega_\rho, \\ \partial_n z_\rho = S z_\rho + \sigma & on \ \ \Gamma, \\ \partial_n z_\rho = 0 & on \ \ \Sigma_\rho. \end{cases}$$

There exist $\rho_2 > 0$ and a constant $c > 0$ independent of ρ, f and σ such that for all $\rho < \rho_2$

$$\|z_\rho\|_{1,\Omega_\rho} \leq c\|f\|_{0,\Omega_\rho} + c\|\sigma\|_{-\frac{1}{2},\Gamma}.$$

This result remains true if we replace S by S^* and k by $\bar{k}$.

Combining Lemma 4 and Lemma 5, we obtain the following result.

Lemma 6 *Consider $\sigma \in H^{-1/2}(\Gamma)$, $\rho \geq 0$, $g \in \mathcal{T}(\Sigma_\rho)$, $f \in L^2(\Omega_\rho)$ and let $z_\rho \in H^1(\Omega_\rho)$ be the solution of the problem*

$$\begin{cases} \Delta z_\rho + k^2 z_\rho = f & in \ \ \Omega_\rho, \\ \partial_n z_\rho = S z_\rho + \sigma & on \ \ \Gamma, \\ \partial_n z_\rho = g & on \ \ \Sigma_\rho. \end{cases} \tag{36}$$

There exist some constants independent of ρ, σ, f and g such that for all ρ sufficiently small

$$\begin{aligned} \|z_\rho\|_{0,\Omega_\rho} &\leq c\rho^{\frac{N}{2}+1}\|g(\rho x)\|_{\mathcal{T}(\Sigma)} + c\|f\|_{0,\Omega_\rho} + c\|\sigma\|_{-\frac{1}{2},\Gamma}, \\ |z_\rho|_{1,\Omega_\rho} &\leq c\rho^{\frac{N}{2}}\|g(\rho x)\|_{\mathcal{T}(\Sigma)} + c\|f\|_{0,\Omega_\rho} + c\|\sigma\|_{-\frac{1}{2},\Gamma}, \\ \|z_\rho\|_{1,D_R} &\leq c\rho^{\frac{N}{2}+1}\|g(\rho x)\|_{\mathcal{T}(\Sigma)} + c\|f\|_{0,\Omega_\rho} + c\|\sigma\|_{-\frac{1}{2},\Gamma}. \end{aligned}$$

This result remains true if we replace S by S^* and k by $\bar{k}$.

8.2 Proof of Theorem 2 (Topological asymptotic for a hole)

We shall successively prove that $\mathcal{E}_i(\rho) = o(\rho^N)$ for $i = 1, ..., 6$.

(1) We set $e_\rho = p_\rho - w_\rho$. We have, using basically the fact that u_0 is of class $\mathcal{C}^\infty$ in the vicinity of the origin,

$$
\begin{aligned}
|\mathcal{E}_1(\rho)| &= \left| \int_{\omega_\rho} \overline{\nabla l_\rho^{e_\rho}}.\nabla u_0 dx \right| \\
&\le |l_\rho^{e_\rho}|_{1,\omega_\rho} |u_0|_{1,\omega_\rho} \\
&\le c\rho^{N-1} |l_\rho^{e_\rho}(\rho x)|_{1,\omega} \|u_0\|_{\mathcal{C}^1(\omega_\rho)} \\
&\le c\rho^{N-1} |l^{e_\rho(\rho x)}|_{1,\omega} \\
&\le c\rho^{N-1} \|e_\rho(\rho x)\|_{H^{\frac{1}{2}}(\Sigma)/\mathbb{C}}.
\end{aligned}
$$

We recall the definition of the quotient norm:

$$
\|u\|_{H^{\frac{1}{2}}(\Sigma)/\mathbb{C}} = \inf_{v-u=C^{st}} \|v\|_{H^{\frac{1}{2}}(\Sigma)}. \tag{37}
$$

Yet, the function e_ρ solves

$$
\begin{cases}
\Delta e_\rho + \bar{k}^2 e_\rho = \bar{k}^2 p_\rho & \text{in } \Omega_\rho, \\
\partial_n e_\rho = -\nabla v_0(0).\mathbf{n} + \partial_n v_0 & \text{on } \Sigma_\rho, \\
\partial_n e_\rho = S^* e_\rho + \partial_n p_\rho - S^* p_\rho & \text{on } \Gamma.
\end{cases}
$$

Hence, by Lemma 6,

$$
\begin{aligned}
|e_\rho|_{1,\Omega_\rho} &\le c\rho^{\frac{N}{2}} \| -\nabla v_0(0).\mathbf{n} + \partial_n v_0(\rho x)\|_{-\frac{1}{2},\Sigma} + c\|\bar{k}^2 p_\rho\|_{0,\Omega_\rho} \\
&\quad + c\|\partial_n p_\rho - S^* p_\rho\|_{-\frac{1}{2},\Gamma} \\
&\le c\rho^{\frac{N}{2}} \| -\nabla v_0(0).\mathbf{n} + \partial_n v_0(\rho x)\|_{-\frac{1}{2},\Sigma} + c\|p_\rho\|_{0,\Omega_\rho} + c\|p_\rho\|_{1,D_R}.
\end{aligned}
$$

Using the interior regularity theorem, we obtain that there exists $\rho_3 > 0$ such that $v_0 \in H^3(\omega_{\rho_3}) \subset \mathcal{C}^1(\omega_{\rho_3})$. Thus, we have

$$
\lim_{\rho \to 0} \| -\nabla v_0(0).\mathbf{n} + \partial_n v_0(\rho x)\|_{-\frac{1}{2},\Sigma} = 0. \tag{38}
$$

Thanks to Lemma 4,

$$
\|p_\rho\|_{0,\Omega_\rho} + \|p_\rho\|_{1,D_R} \le c\rho^{\frac{N}{2}+1} \|\nabla v_0(0).\mathbf{n}\|_{-\frac{1}{2},\Sigma} \le c\rho^{\frac{N}{2}+1}.
$$

Hence

$$
|e_\rho|_{1,\Omega_\rho} = o(\rho^{\frac{N}{2}}).
$$

Using successively the trace theorem, the equivalence of the norm and the semi-norm in $H^1/\mathbb{C}$ and a change of variable and denoting by B some ball containing ω, we obtain that

$$
\|e_\rho(\rho x)\|_{H^{\frac{1}{2}}(\Sigma)/\mathbb{C}} \le c\|e_\rho(\rho x)\|_{H^1(B\setminus\bar{\omega})/\mathbb{C}} \le c|e_\rho(\rho x)|_{1,B\setminus\bar{\omega}} \le c\rho^{1-\frac{N}{2}} |e_\rho|_{1,\Omega_\rho},
$$

from which we deduce that $\mathcal{E}_1(\rho) = o(\rho^N)$.

(2) The fact that, in the vicinity of the origin, L is continuous and u_0 is of class $\mathcal{C}^\infty$ yields directly $|\mathcal{E}_2(\rho)| \le c\rho^{N+1}$.

(3) We have

$$\mathcal{E}_3(\rho) = k^2 u_0(0) \int_{\omega_\rho} (\overline{v_0} - \overline{v_0(0)})dx.$$

Since v_0 is of class $\mathcal{C}^1$ in the vicinity of the origin, we obtain immediately with the help of a Taylor expansion that $|\mathcal{E}_3(\rho)| \leq c\rho^{N+1}$.

(4) We get by a change of variable

$$|\mathcal{E}_4(\rho)| = k^2 \rho^N \left| \int_\omega u_0(\rho x) \overline{l^{w_\rho(\rho x)}} dx \right| \leq c\rho^N \| l^{w_\rho(\rho x)} \|_{0,\omega}.$$

The elliptic regularity, the trace theorem and a change of variable bring successively

$$\begin{aligned} |\mathcal{E}_4(\rho)| &\leq c\rho^N \|w_\rho(\rho x)\|_{\frac{1}{2},\Sigma} \\ &\leq c\rho^N \|w_\rho(\rho x)\|_{1,\frac{1}{\rho}\Omega_\rho} \\ &\leq c\rho^N \left(\rho^{-\frac{N}{2}} \|w_\rho\|_{0,\Omega_\rho} + \rho^{1-\frac{N}{2}} |w_\rho|_{1,\Omega_\rho} \right). \end{aligned}$$

Then, Lemma 6 and the fact that v_0 is of class $\mathcal{C}^1$ in the vicinity of the origin furnish

$$|\mathcal{E}_4(\rho)| \leq c\rho^{N+1} \|\partial_n v_0(\rho x)\|_{-\frac{1}{2},\Sigma} \leq c\rho^{N+1}.$$

(5) We have

$$|\mathcal{E}_5(\rho)| \leq \rho^{N-1} \|\partial_n v_0(\rho x) - \nabla v_0(0).\mathbf{n}\|_{-\frac{1}{2},\Sigma} \|u_0(\rho x) - u_0(0)\|_{\frac{1}{2},\Sigma}.$$

Equation (38) and the regularity of u_0 near the origin yield $\mathcal{E}_5(\rho) = o(\rho^N)$.

(6) We have

$$|\mathcal{E}_6(\rho)| \leq \rho^{N-1} \|\lambda\|_{-\frac{1}{2},\Sigma} \|u_0(\rho y) - u_0(0) - \nabla u_0(0).\rho y\|_{\frac{1}{2},\Sigma}.$$

With the help of a Taylor expansion, we derive $|\mathcal{E}_6(\rho)| \leq c\rho^{N+1}$, which completes the proof of the theorem. □

8.3 Proof of Theorem 3 (Topological asymptotic for a crack)

We have here to prove that $\mathcal{E}_i(\rho) = o(\rho^N)$ for all $i = 1, ..., 3$.

(1) Setting $e_\rho = p_\rho - w_\rho$, we have

$$\begin{aligned} |\mathcal{E}_1(\rho)| &\leq \rho^{N-1} \|\partial_n u_0(\rho x)\|_{(H^{\frac{1}{2}}_{00}(\Sigma))'} \|[e_\rho(\rho x)]\|_{H^{\frac{1}{2}}_{00}(\Sigma)} \\ &\leq c\rho^{N-1} \|[e_\rho(\rho x)]\|_{H^{\frac{1}{2}}_{00}(\Sigma)}. \end{aligned}$$

The function e_ρ solves

$$\begin{cases} \Delta e_\rho + \bar{k}^2 e_\rho = \bar{k}^2 p_\rho & \text{in } \Omega_\rho, \\ \partial_n e_\rho = 0 & \text{on } \Sigma_\rho, \\ \partial_n e_\rho = S^* e_\rho + \partial_n p_\rho - S^* p_\rho & \text{on } \Gamma. \end{cases}$$

Therefore, Lemma 5 yields

$$\begin{aligned} \|e_\rho\|_{1,\Omega_\rho} &\leq c\|\bar{k}^2 p_\rho\|_{0,\Omega_\rho} + c\|\partial_n p_\rho - S^* p_\rho\|_{-\frac{1}{2},\Gamma} \\ &\leq c\|p_\rho\|_{0,\Omega_\rho} + c\|p_\rho\|_{1,D_R}. \end{aligned}$$

Yet, according to Lemma 4,

$$\|p_\rho\|_{0,\Omega_\rho} + \|p_\rho\|_{1,D_R} \leq c\rho^{\frac{N}{2}+1} \|\partial_n u_0(\rho x)\|_{(H_{00}^{\frac{1}{2}}(\Sigma))'} \leq c\rho^{\frac{N}{2}+1}.$$

Thus,

$$\|e_\rho\|_{1,\Omega_\rho} \leq c\rho^{\frac{N}{2}+1}.$$

Besides, we have the estimates

$$\|[e_\rho(\rho x)]\|_{H_{00}^{\frac{1}{2}}(\Sigma)} \leq c\|e_\rho(\rho x)\|_{H^1(B\setminus\Sigma)/\mathbb{C}} \leq c|e_\rho(\rho x)|_{1,B\setminus\Sigma} \leq c\rho^{1-\frac{N}{2}} |e_\rho|_{1,\Omega_\rho}.$$

It follows that $|\mathcal{E}_1(\rho)| \leq c\rho^{N+1}$.

(2) We have

$$\begin{aligned} |\mathcal{E}_2(\rho)| &\leq \rho^N \|\partial_n u_0(\rho x)\|_{(H_{00}^{\frac{1}{2}}(\Sigma))'} \|\mu_\rho - \mu\|_{H_{00}^{\frac{1}{2}}(\Sigma)} \\ &\leq c\rho^N \|\mu_\rho - \mu\|_{H_{00}^{\frac{1}{2}}(\Sigma)}. \end{aligned}$$

By continuity of the operator T, it comes

$$\|\mu_\rho - \mu\|_{H_{00}^{\frac{1}{2}}(\Sigma)} \leq c\|\partial_n v_0(\rho x) - \nabla v_0(0).\mathbf{n}\|_{(H_{00}^{\frac{1}{2}}(\Sigma))'}.$$

Hypothesis 7 guarantees that v_0 is of class $\mathcal{C}^1$ in the vicinity of the origin. Thus,

$$\lim_{\rho\to 0} \|\partial_n v_0(\rho x) - \nabla v_0(0).\mathbf{n}\|_{(H_{00}^{\frac{1}{2}}(\Sigma))'} = 0 \tag{39}$$

and $\mathcal{E}_2(\rho) = o(\rho^N)$.

(3) We have

$$|\mathcal{E}_3(\rho)| \leq \rho^N \|\partial_n u_0(\rho x) - \nabla u_0(0).\mathbf{n}\|_{(H_{00}^{\frac{1}{2}}(\Sigma))'} \|\mu\|_{H_{00}^{\frac{1}{2}}(\Sigma)}.$$

Next, as u_0 is of class $\mathcal{C}^\infty$ in the vicinity of the origin, we derive that $|\mathcal{E}_3(\rho)| \leq c\rho^{N+1}$, which ends up the proof of the theorem. □

8.4 Proof of Theorem 4 (Examples of cost functions)

We start with a preliminary lemma. The proof consists in applying Lemma 6 to the function $u_\rho - u_0$ restricted to Ω_ρ.

Lemma 7 *We have the estimates*

$$\|u_\rho - u_0\|_{0,\Omega_\rho} = O(\rho^{\frac{N}{2}+1}) \quad \textit{and} \quad \|u_\rho - u_0\|_{1,D_R} = O(\rho^{\frac{N}{2}+1}).$$

We now turn to the proof of Theorem 4.

(1) For the first category of cost functions, the result is an immediate application of Lemma 7.
(2) For the second example, we only present the case of a perforation. The case of a crack can be treated in a similar way. We have

$$J_\rho(u_\rho) - J_0(u_0) = \int_{\Omega_\rho} (|u_\rho - u_d|^2 - |u_0 - u_d|^2) dx - \int_{\omega_\rho} |u_0 - u_d|^2 dx$$
$$= \int_{\Omega_\rho} \left[|u_\rho - u_0|^2 + 2\Re((\overline{u_0 - u_d})(u_\rho - u_0)) \right] dx - \int_{\omega_\rho} |u_0 - u_d|^2 dx.$$

Lemma 6, the regularity of u_0 and u_d near the origin and the fact that

$$L_\rho(u) = \int_{\Omega_\rho} 2(\overline{u_0 - u_d}) u dx \qquad \forall u \in H^1(\Omega_\rho)$$

yield

$$J_\rho(u_\rho) - J_0(u_0) = \Re L_\rho(u_\rho - u_0) - \rho^N |\omega| |u_0(0) - u_d(0)|^2 dx + o(\rho^N),$$

which completes the proof.

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